Image formation in the paraxial approximation

1- Introduction

- Simple systems do not usually give perfect images
- Calculating the path of many rays using Snell’s law is tedious

Paraxial approximation (rays near axis) gives:

- Perfect images for any optical system: lens, mirror…
- Simple expressions for images positions and sizes
- Convenient reference from which to measure departures from perfection (aberrations)

Image formation in the paraxial approximation

1- Introduction

2- The paraxial (or gaussian) approximation :

- ↗ Centered optical systems (optical axis)
- ↗ Small pencil of rays near the optical axis
  - ↗ small angles of incidence
  - ↗ small aperture (α)
  - ↗ small field (y)
  - ↗ spherical surfaces replaced by tangent planes

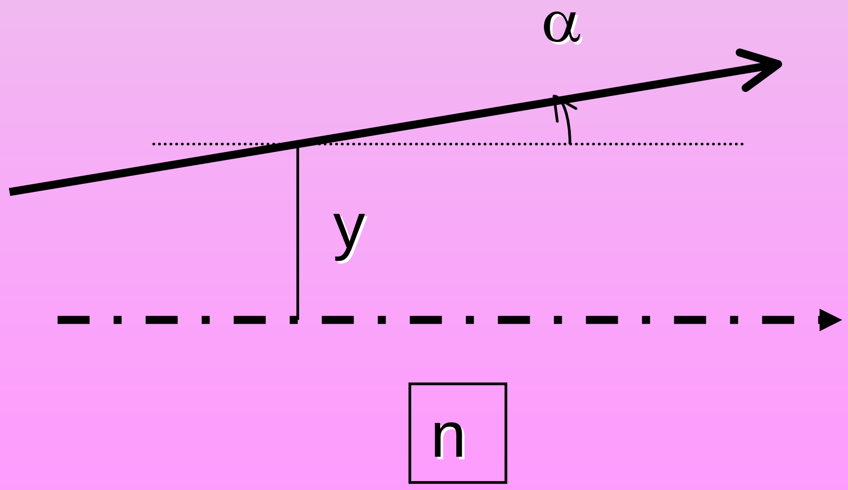
Image formation in the paraxial approximation

1- Introduction

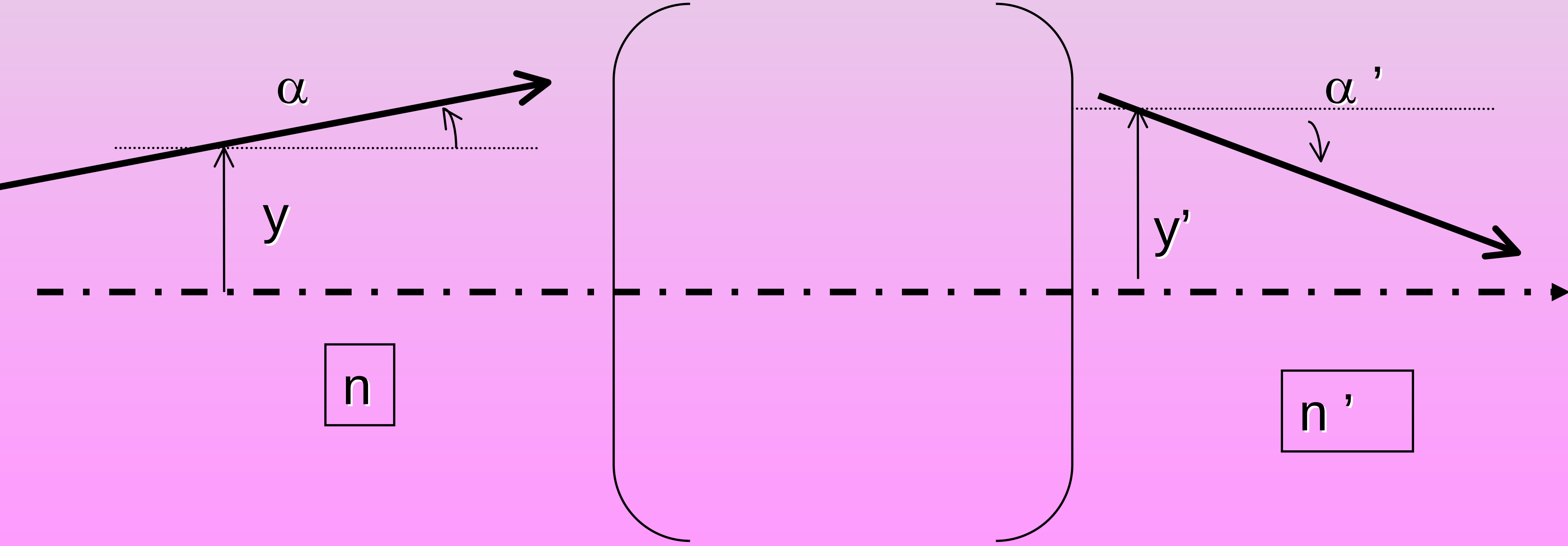
2- The paraxial approximation

3- Ray vector :

$$\begin{bmatrix} y \\ n\alpha \end{bmatrix}$$



$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} ab \\ cd \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$



Question :   how to determine this *abcd* matrix?

Image formation in the paraxial approximation

1- Introduction

2- The paraxial approximation

3- Ray vector

4- Homogeneous medium : the translation matrix

$$\begin{bmatrix} y' \\ n\alpha \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{d}{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

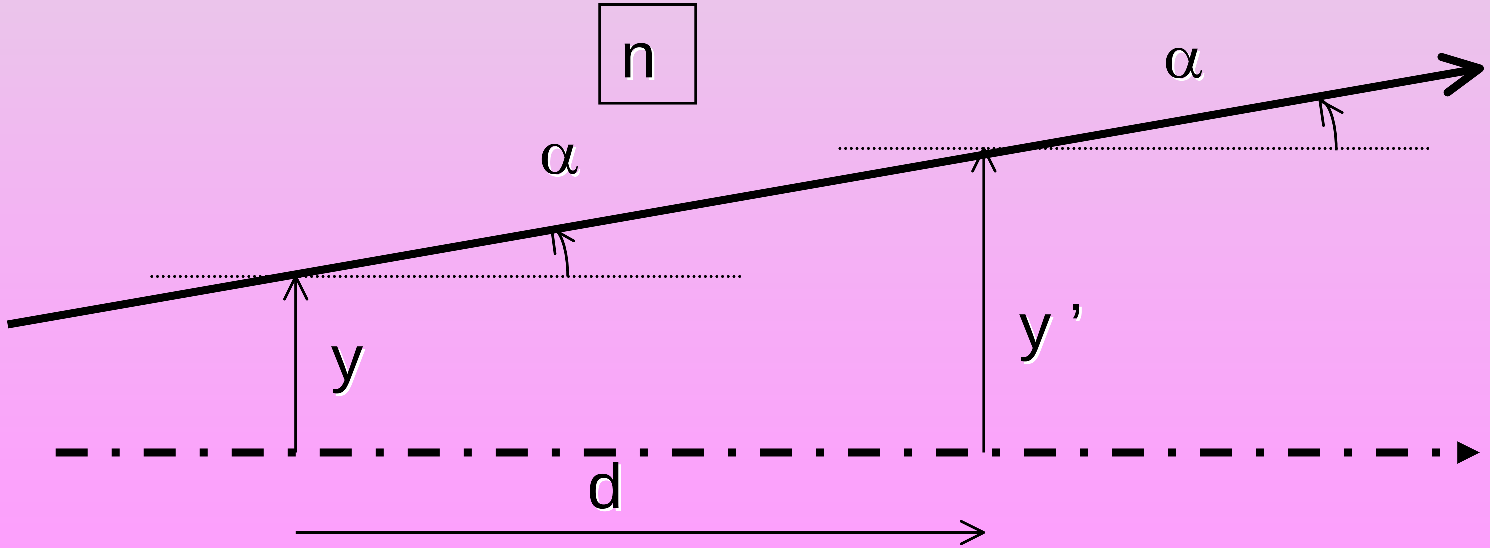


Image formation in the paraxial approximation

1- Introduction

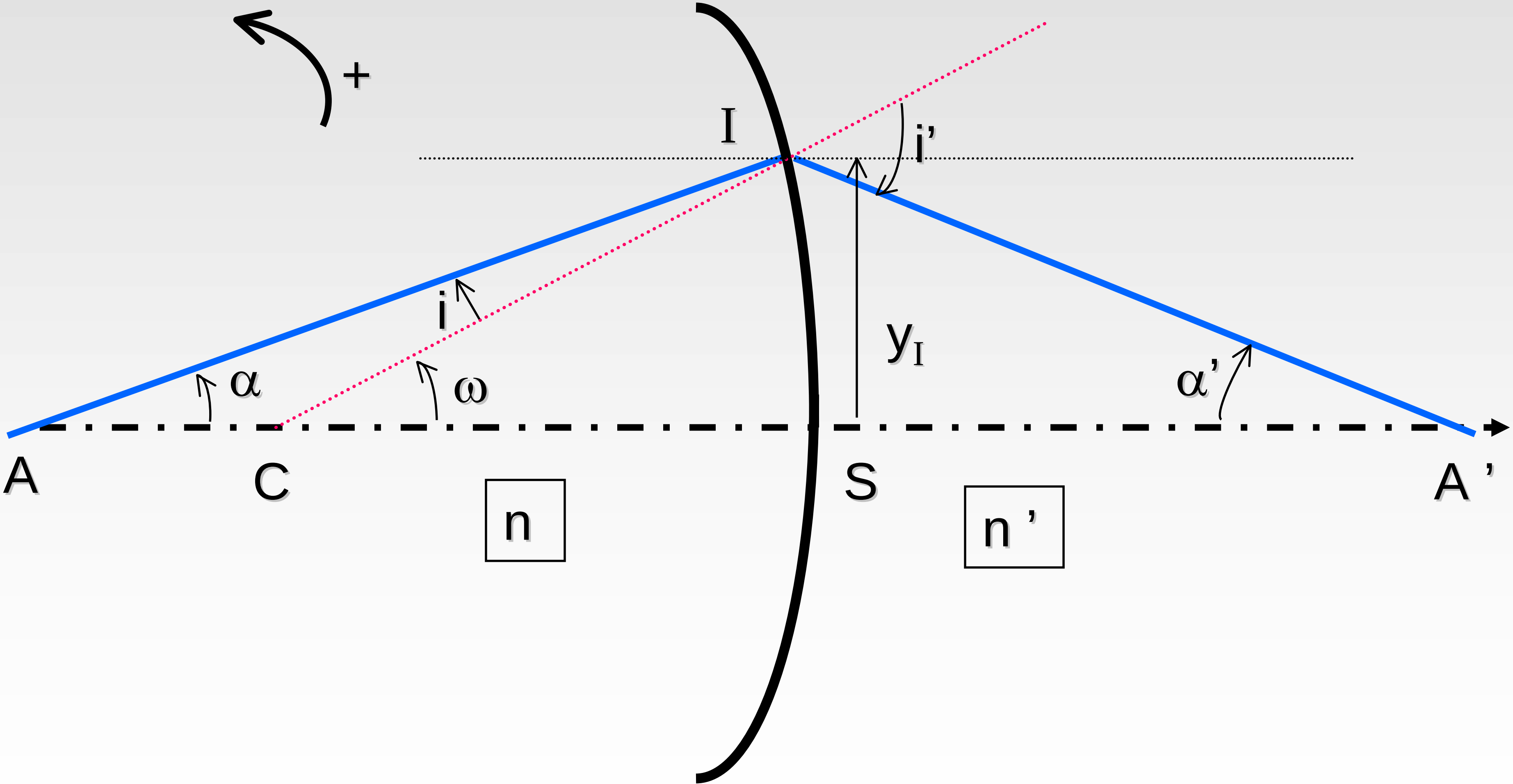
2- The paraxial approximation

3- Ray vector

4- Homogenous medium : the translation matrix

5- Matrix for one refractive surface

Spherical surface



Matrix for an air-glass spherical surface

- 5- *a*- *Refraction Matrix :*

$$\begin{bmatrix} y_I \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{n'-n}{SC} & 1 \end{bmatrix} \begin{bmatrix} y_I \\ n\alpha \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}$$

Power of the surface : $P = \frac{n'-n}{SC}$

Spherical surface in the paraxial approximation

- b : *Gaussian formula :*

$$\frac{n'}{SA'}-\frac{n}{SA}=\frac{n'-n}{SC}$$

- c : *Focal points and focal lengths*

$$\overline{SF}=-\frac{n}{n'-n}\overline{SC}$$

$$f=-\frac{n}{P}$$

1st fl

$$\overline{SF'}=\frac{n'}{n'-n}\overline{SC}$$

$$f=\frac{n'}{P}$$

2nd fl

Example : the eye

Air-water surface :

- Power : **60 δ** (diopters=1m^{-1})
- Surface curvature ?
- First and second focal lengths ?

Spherical surface in the paraxial approximation

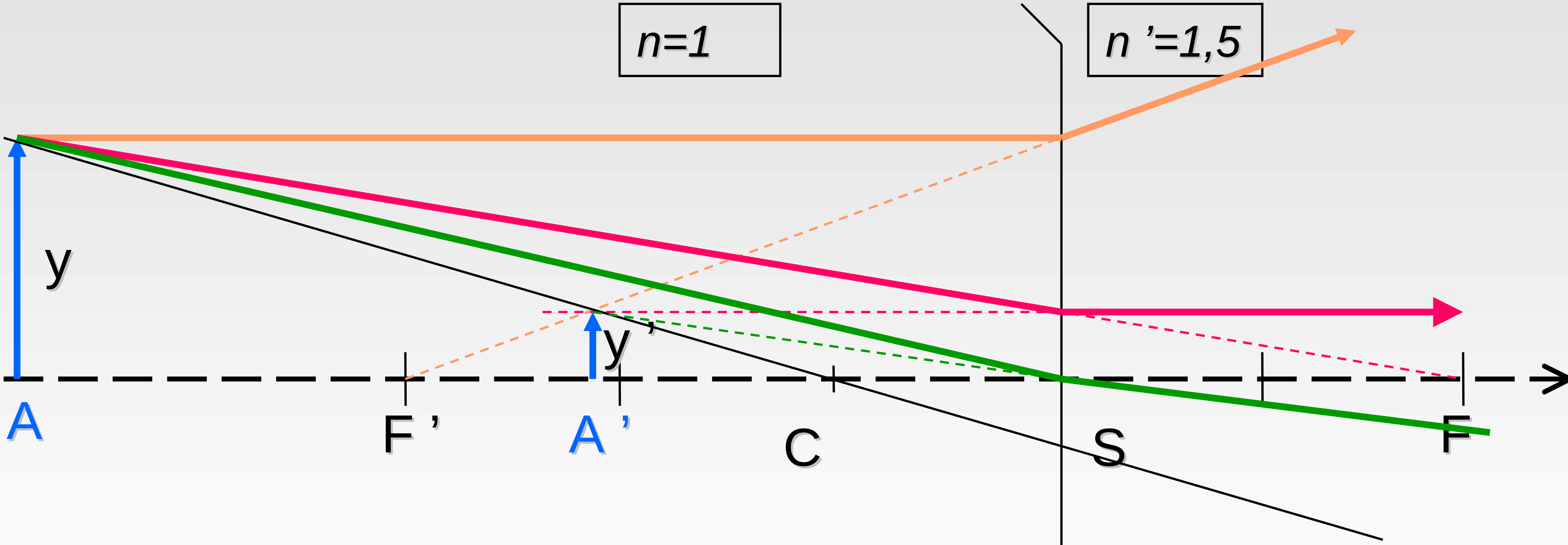
d : *Transverse magnification*

Newton’s formula

Lagrange invariant

$$g_y = \frac{y'}{y} = \frac{\overline{F'A'}}{\overline{F'S}} = -\frac{\overline{F'A'}}{f'}$$

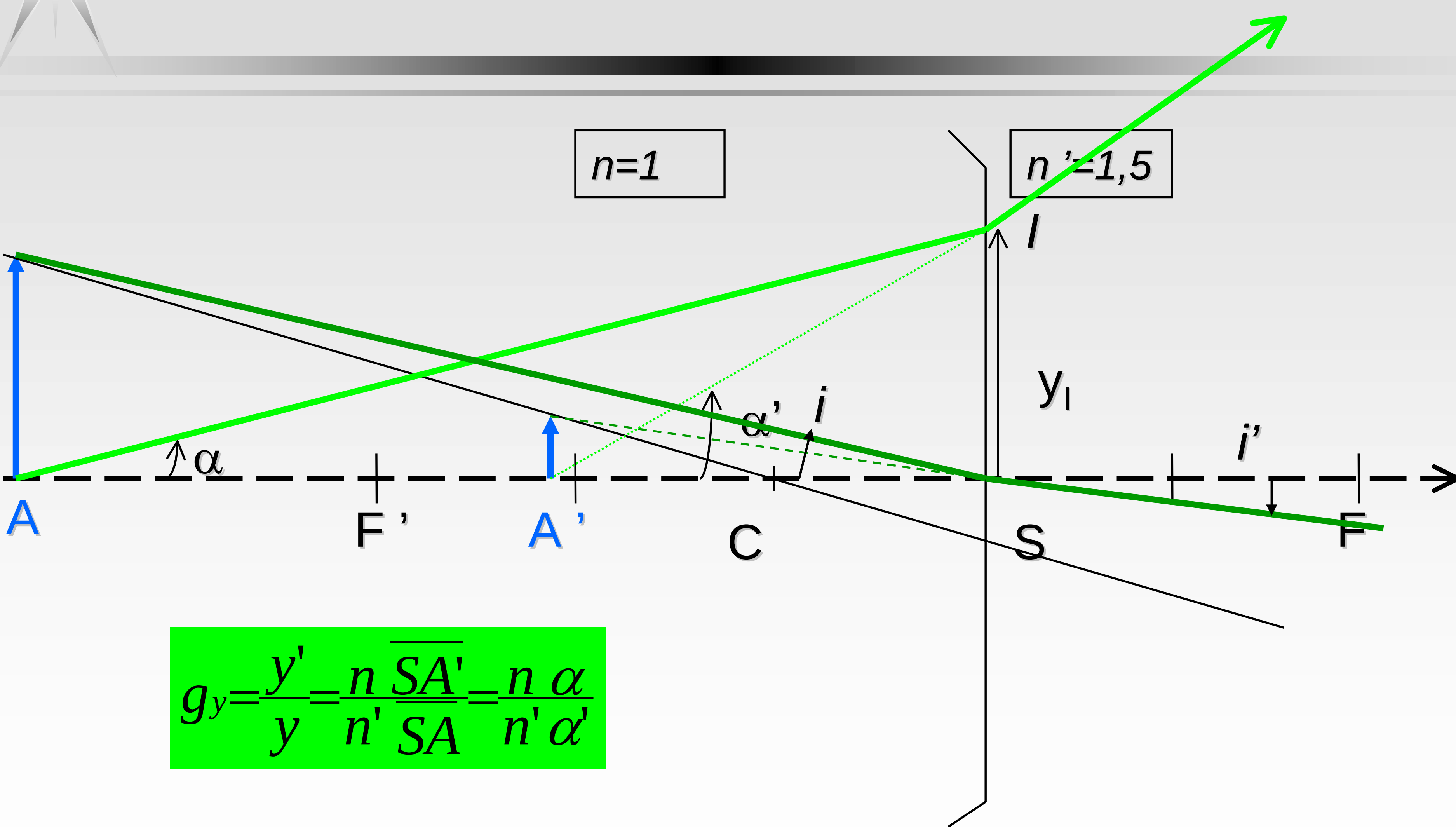
$$g_y = \frac{y'}{y} = \frac{\overline{FS}}{\overline{FA}} = -\frac{f}{\overline{FA}}$$



$$g_y = \frac{y'}{y} = \frac{n}{n'}\frac{\overline{SA'}}{\overline{SA}}$$

$$g_y = \frac{y'}{y} = \frac{\overline{CA'}}{\overline{CA}}$$

Lagrange formula



$$g_y = \frac{y'}{y} = \frac{n\,\overline{SA'}}{n'\,\overline{SA}} = \frac{n\,\alpha}{n'\,\alpha'}$$

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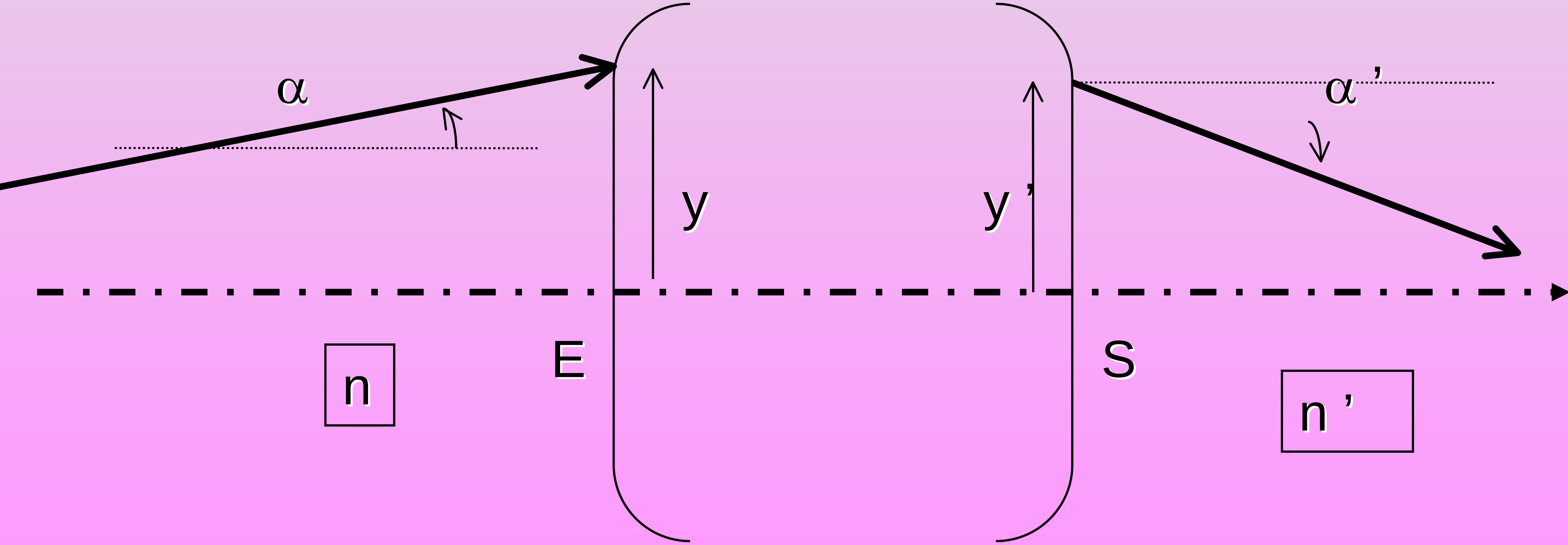
5- Matrix for one refractive surface

6- General matrix for any centered system

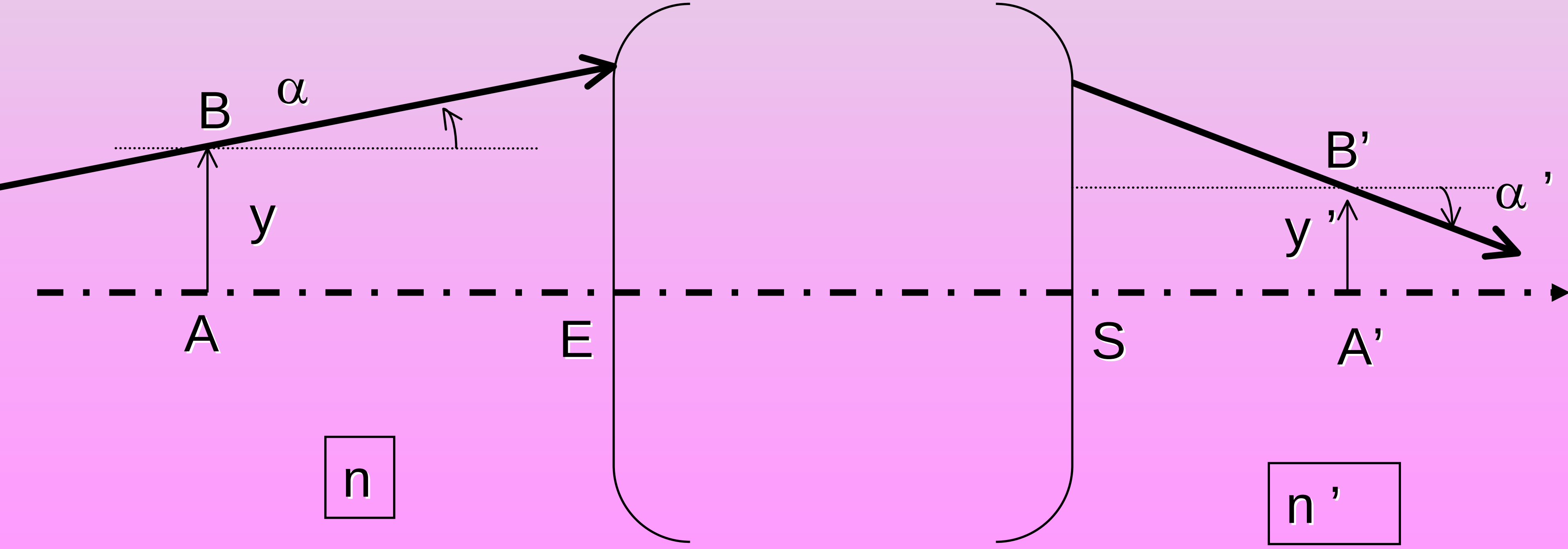
Matrix from E (first refractive surface) to S (last surface)

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{E\to S} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$



Change of references for the incident and emerging rays

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} 1 & \overline{SA'}/n' \\ 0 & 1 \end{bmatrix}_{S\to A'} \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{E\to S} \begin{bmatrix} 1 & \overline{AE}/n \\ 0 & 1 \end{bmatrix}_{A\to E} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$



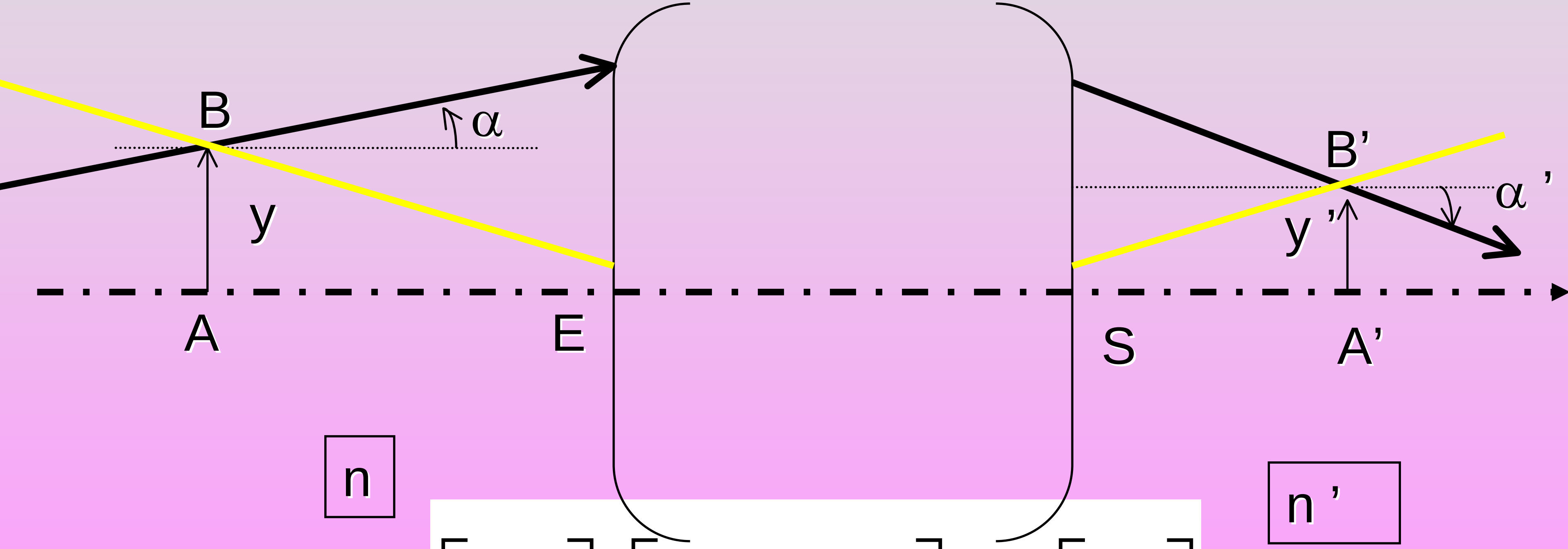
Matrix between two arbitrary points A and A’ for any optical system :

$$\begin{bmatrix} a + c\frac{\overline{SA'}}{n'} & a\frac{\overline{AE}}{n} + b + d\frac{\overline{SA'}}{n'} + c\frac{\overline{AE}}{n}\frac{\overline{SA'}}{n'} \\ c & c\frac{\overline{AE}}{n} + d \end{bmatrix}$$

independent on A & A’ ⇒ Power : P=-c

What happens if A and A’ are conjugate points ?

If B and B’ are conjugate points: y’ should not depend on α



$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a' & b'=0 \\ c' & d' \end{bmatrix}_{A\rightarrow A'} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

A and A ’ conjugate points ⇔ *b ’* = 0

A and A ’ conjugate points ⇔ *b* ’ = 0

Gaussian formula for 2 conjugate planes in A and A’ for any system:

$$\begin{bmatrix} a + c\frac{\overline{SA'}}{n'} & a\frac{\overline{AE}}{n} + b + d\frac{\overline{SA'}}{n'} + c\frac{\overline{AE}}{n}\frac{\overline{SA'}}{n'} \\ c & c\frac{\overline{AE}}{n} + d \end{bmatrix}$$

= 0

General expression for the position of the image :

$$\frac{\overline{SA'}}{n'} = \frac{a\frac{\overline{EA}}{n} - b}{-c\frac{\overline{EA}}{n} + d}$$

A and A ’ conjugate points ⇔ *b* ’ = 0

Corresponding transverse magnification :

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a + c\dfrac{\overline{SA'}}{n'} & 0 \\ c & c\dfrac{\overline{AE}}{n} + d \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

g_y

$(n'/n)g_\alpha$

$$\frac{\overline{SA'}}{n'} = \frac{a\dfrac{\overline{EA}}{n} - b}{-c\dfrac{\overline{EA}}{n} + d}$$

$$(n'/n)\, g_y\, g_\alpha = 1$$

$$(\forall\ y) \qquad g_y = \frac{y'}{y} = a' = a + c\frac{\overline{SA'}}{n'}$$

$$(\text{and for } y = 0,\ \text{ray from A}) \qquad \frac{n'\alpha'}{n\alpha} = d' = d - c\frac{\overline{EA}}{n}$$

Very important special case $c=0$: zero Power, Afocal system

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

An incident ray parallel to the axis ($\alpha=0$) comes out parallel to the axis ($\alpha'=0$)

Magnification is identical for any conjugate planes :

$$g_y = \frac{y'}{y} = a$$

$$\frac{n'\alpha'}{n\alpha} = d = \frac{1}{a} = \frac{y}{y'}$$

On the contrary, with a **focal** centered system (**P≠0**), we can obtain **any magnification (!) , g_y** , for one specific pair of conjugate points.

g_y

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a + c\dfrac{\overline{SA'}}{n'} & 0 \\ c & c\dfrac{\overline{AE}}{n} + d \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

$1/g_y$

$$g_y = \frac{y'}{y} = a' = a + c\frac{\overline{SA'}}{n'}$$

$$\frac{\overline{SA'}}{n} = \frac{g_y - a}{c}$$

Corresponding image position

$$\frac{\overline{AE}}{n} = \frac{g_y^{-1} - d}{c}$$

Corresponding object position

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6- Matrix for a lens

7- Matrix for a centered system

8- **Cardinal points for a focal centered system ($P \neq 0$)**

For a converging or diverging system, we can always find two conjugate planes corresponding to a tranverse magnification **+1***. Those planes are called PRINCIPAL PLANES, refered as* **H** *(1st principal plane, in object space) and* **H’** *(2nd principal plane).*

$g_y = 1$

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} a + c\dfrac{\overline{SA'}}{n'} & 0 \\ c & c\dfrac{\overline{AE}}{n} + d \end{bmatrix} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

$1/g_y$

$$g_y = \frac{y'}{y} = a' = a + c\frac{\overline{SA'}}{n'}$$

$$\frac{\overline{SA'}}{n'} = \frac{g_y - a}{c}$$

$$\frac{\overline{AE}}{n} = \frac{g_y^{-1} - d}{c}$$

$$\frac{\overline{SH'}}{n'} = \frac{1 - a}{c}$$

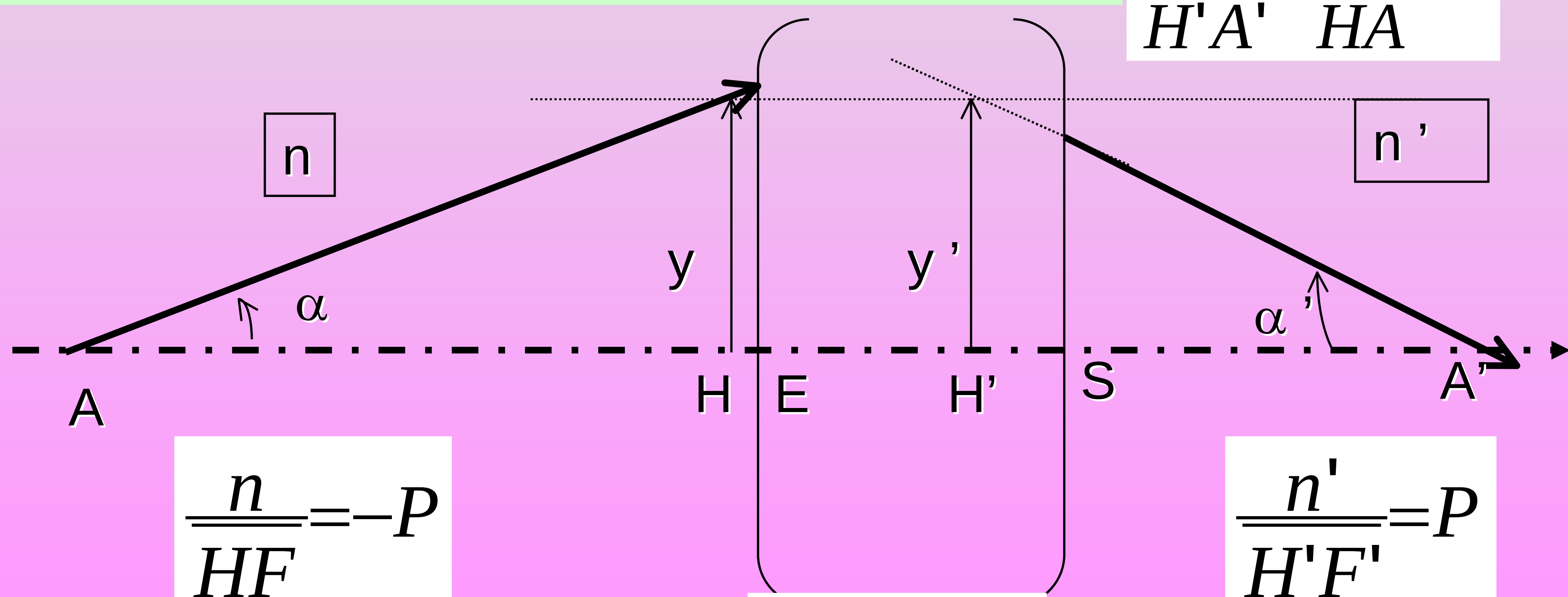
$$\frac{\overline{AH}}{n} = \frac{1 - d}{c}$$

For any focal centered system, the matrix between the 2 principal planes is the same as for the single surface !

$$\begin{bmatrix} y' \\ n'\alpha' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -P & 1 \end{bmatrix}_{H\to H'} \begin{bmatrix} y \\ n\alpha \end{bmatrix}$$

Gaussian formula for the position of the image:

$$\frac{n'}{\overline{H'A'}} - \frac{n}{\overline{HA}} = P$$



$$\frac{n}{\overline{HF}} = -P$$

$$f = -n/P$$

Ray

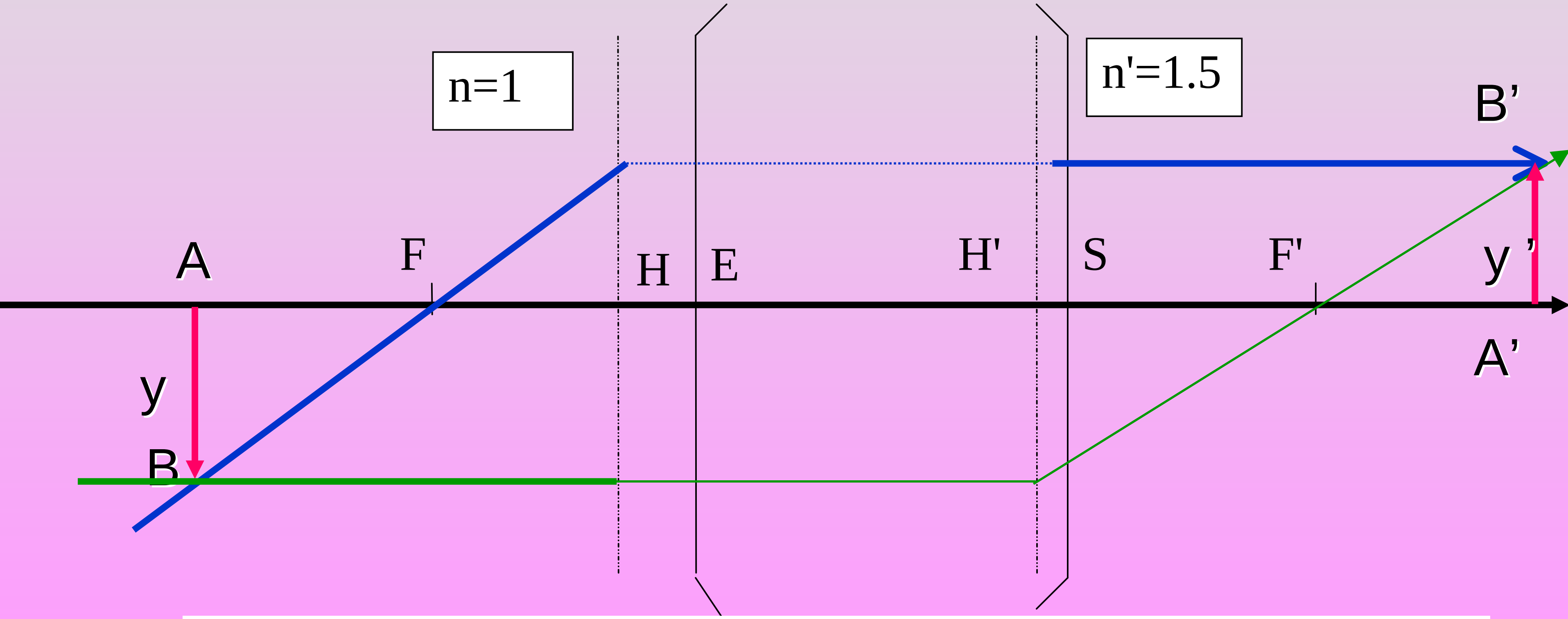
$$f'/f = -n'/n$$

$$\frac{n'}{\overline{H'F'}} = P$$

$$f' = n'/P$$

Construction of specific rays

First principal plane (H): intersection of all incident rays parallel to the optical axis and emerging in F’



Second principal plane (H’): intersection of all incident rays from F and emerging parallel to the optical axis

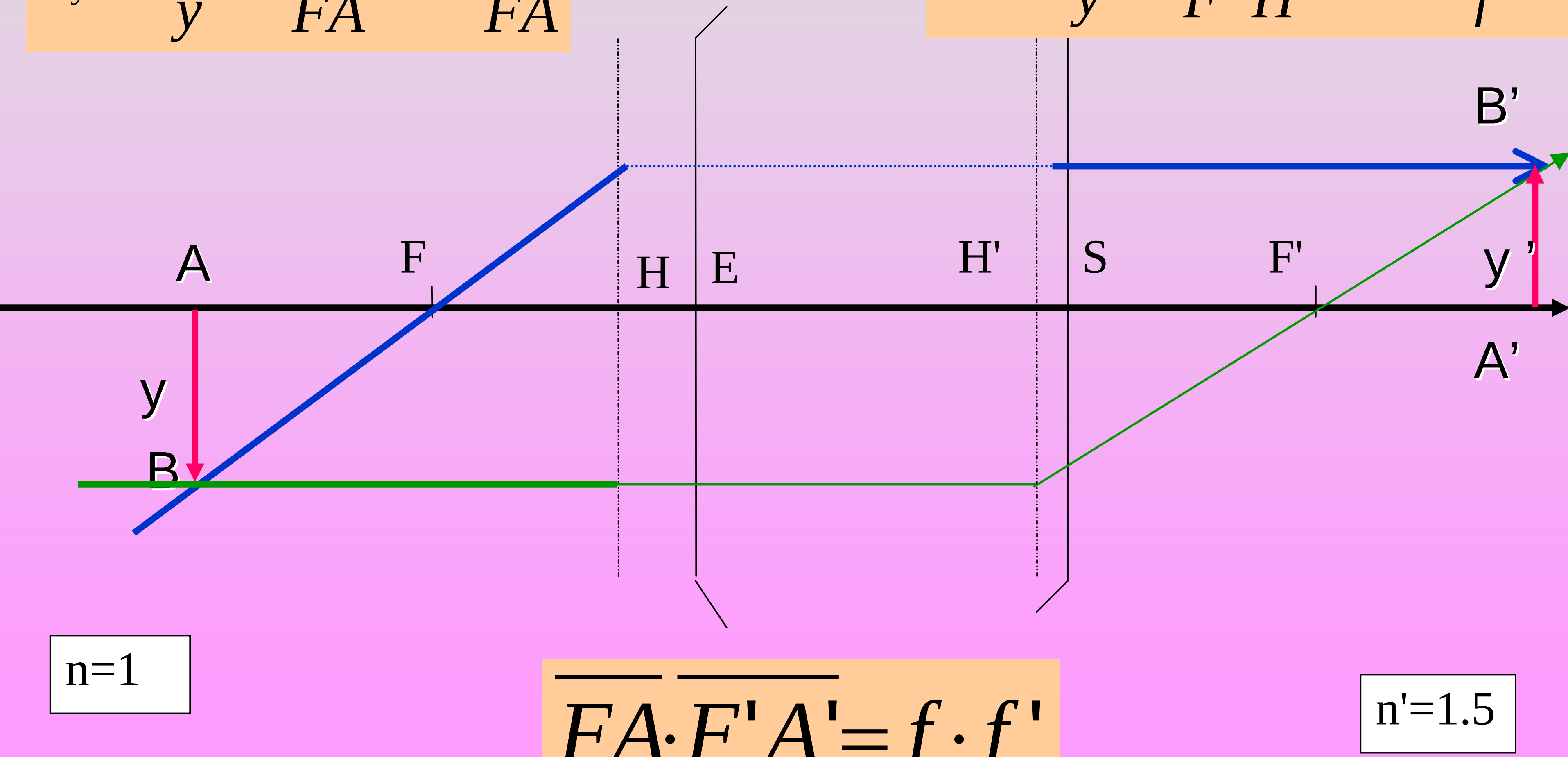
See other constructions on the list « basic exercises on paraxial approximation »

Magnifications

Newtonian formula

$$g_y = \frac{y'}{y} = \frac{\overline{FH}}{\overline{FA}} = -\frac{f}{\overline{FA}}$$

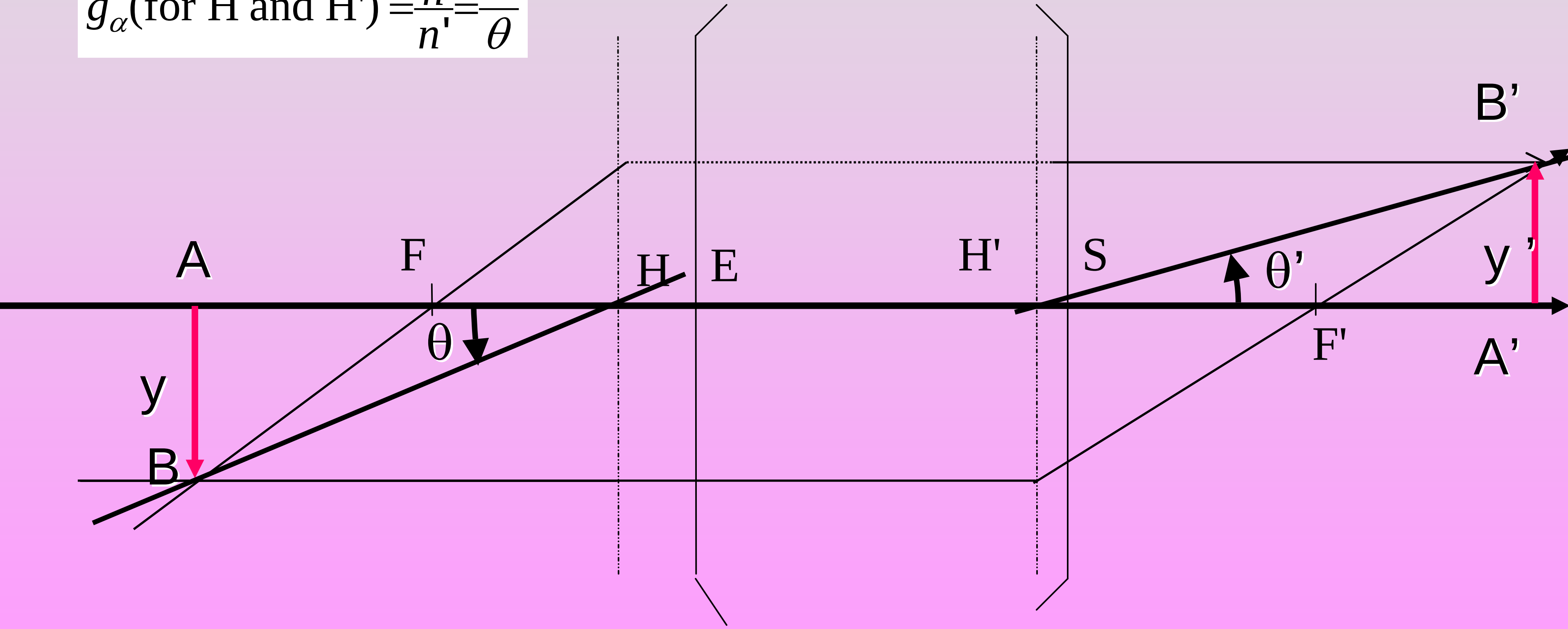
$$g_y = \frac{y'}{y} = \frac{\overline{F'A'}}{\overline{F'H'}} = -\frac{\overline{F'A'}}{f'}$$



$$\overline{FA}\cdot\overline{F'A'} = f\cdot f'$$

Magnification with respect to the principal planes

$$g_\alpha(\text{for H and H'}) = \frac{n}{n'} = \frac{\theta'}{\theta}$$



n=1

$$g_y = \frac{y'}{y} = \frac{n}{n'}\frac{\overline{H'A'}}{\overline{HA}}$$

n'=1.5

Other useful cardinal points

Nodal points: angular magnification =+1

(different from principal points if $n \neq n'$)

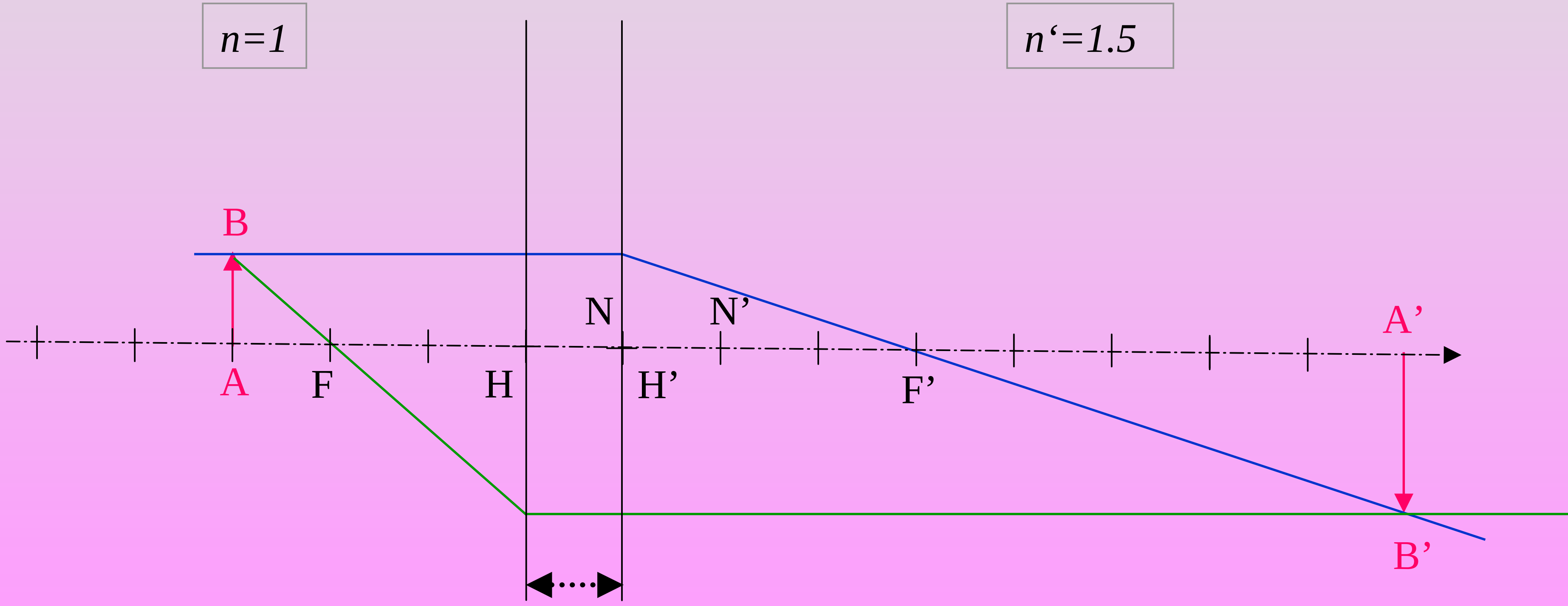
Antiprincipal points: transverse magnification = -1

Antinodal points: angular magnification = -1

Exercises of constructions using foci and principal planes

P=50 diopters

$\overline{HH'}=1\text{cm}$



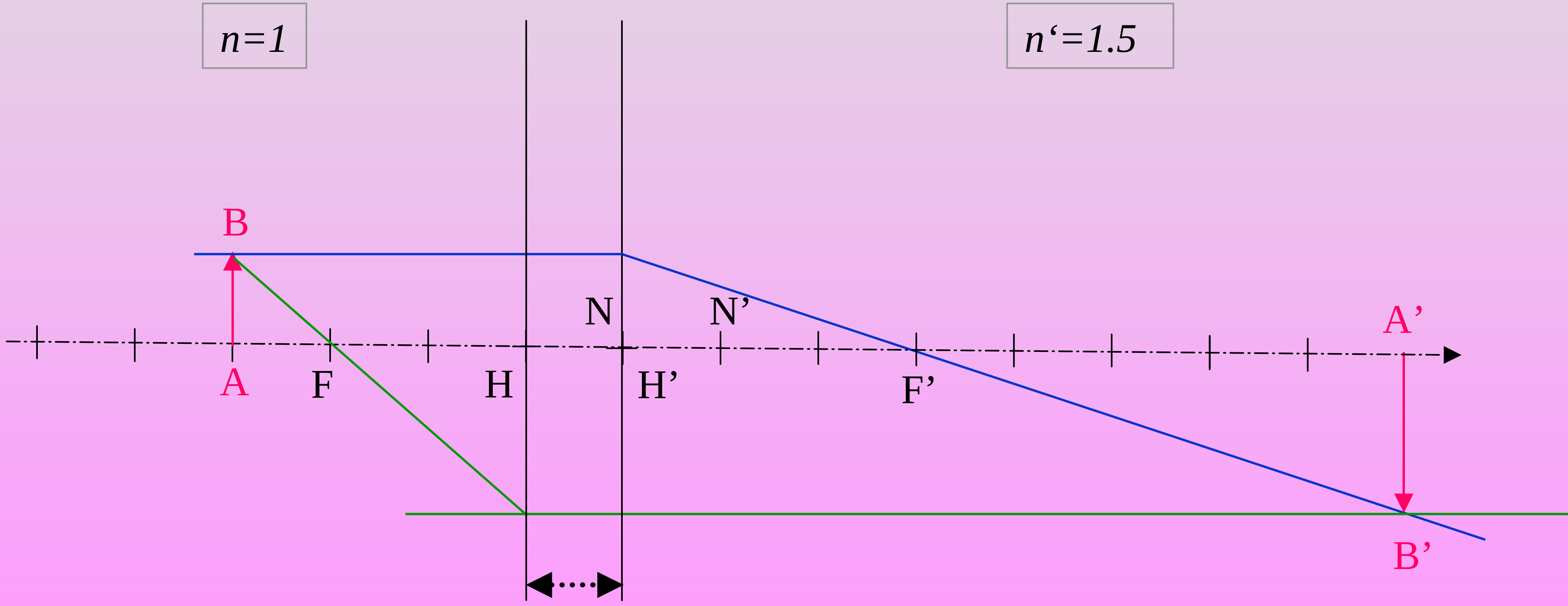
(1) Place F, F', N, N'

(2) Construct the image of A for FA=-1cm

Effect of variation of distance between principal planes

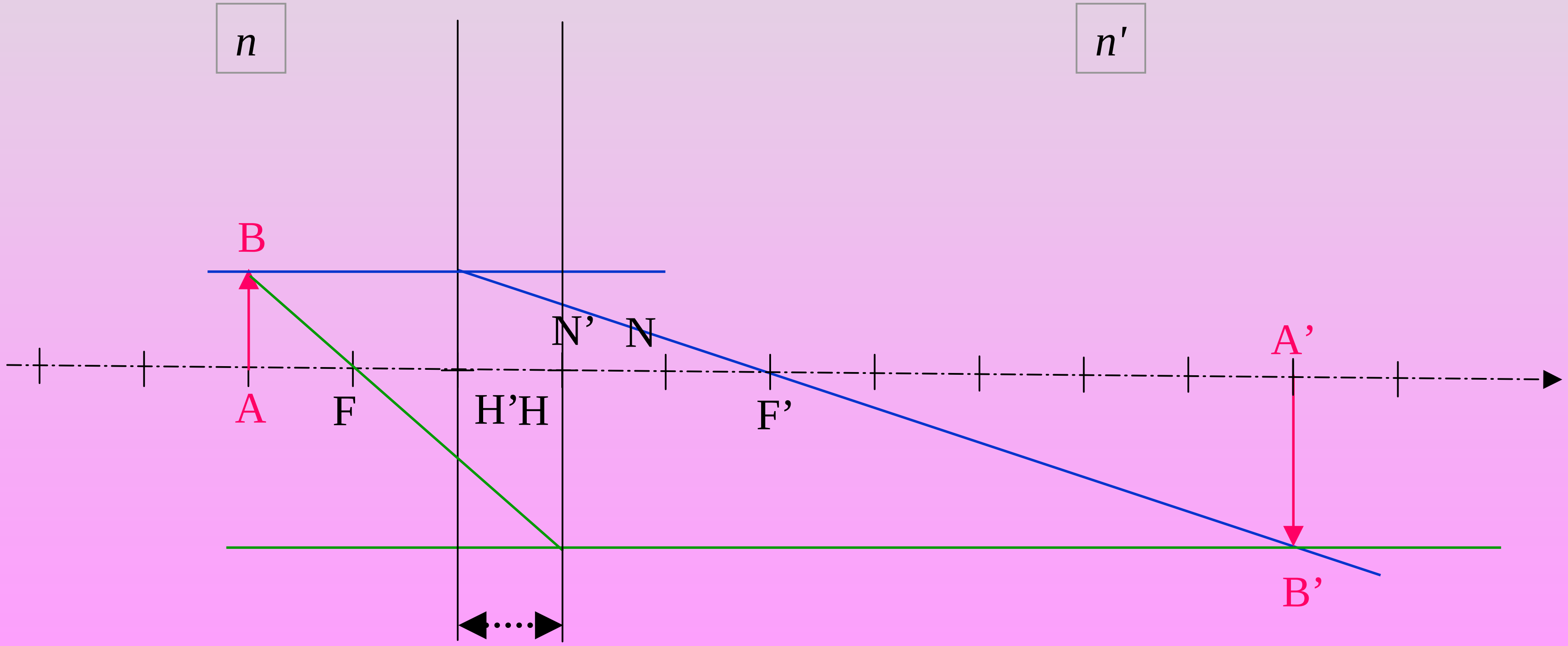
P=50 diopters

$$\overline{HH'}=1\text{cm}$$



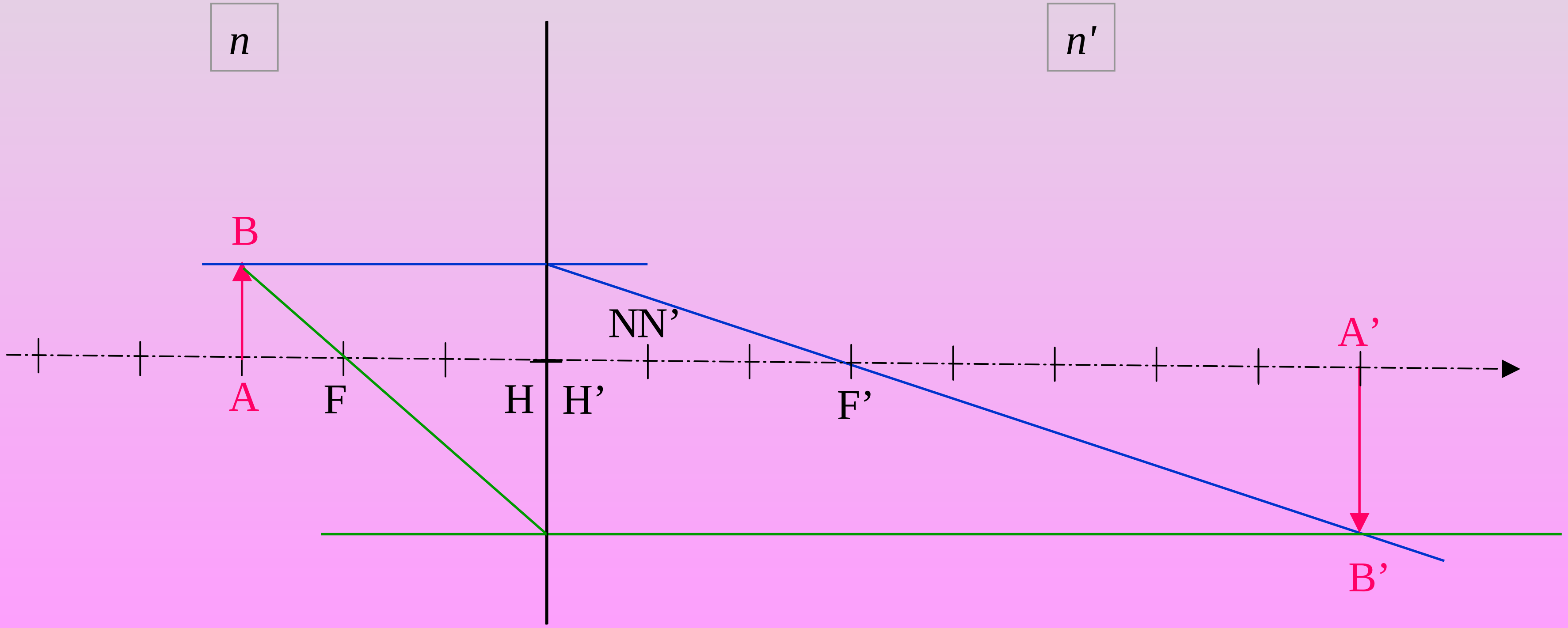
Effect of variation of distance between principal planes

$$\overline{HH'} = -1\text{cm}$$



Effect of variation of distance between principal planes

$$\overline{HH'}=0$$



Equivalent to a single spherical surface if n≠n'

Equivalent to a thin lens if n=n'