CHAPTER II

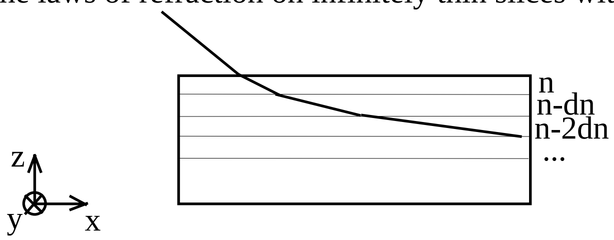
APPLICATION OF FERMAT'S PRINCIPLE

TO INHOMOGENEOUS MEDIA

The goal of this chapter is to present a few properties of light in inhomogeneous media, first as an application of Fermat's principle, second to explain natural phenomena such as mirages or atmospheric refraction, and how gradient-index fibers and lenses work. It is an introduction to the study of these media.

I. Introduction

Let us consider a simple inhomogeneous medium with an index n independent of x and y, and that increases linearly with z. We can try to guess how the rays will propagate in such a medium using the laws of refraction on infinitely thin slices with height dz.

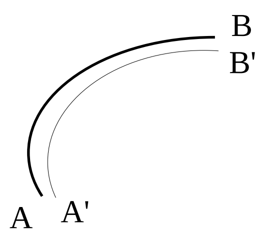


As the index of refraction diminishes, the refracted angle increases and the ray bends. In order to have a more quantitative approach, we will go back to Fermat's principle.

II. Trajectory of a light ray in an inhomogeneous medium.

1) Calculation of the differential of an optical path

Let us consider any light ray between two points A and B in an inhomogeneous medium with index $n(x,y,z)$ and calculate the difference with a very close path A'B'.



The optical path from A to B is given by:

$$L=\int_A^B n\,ds$$

The difference between optical paths A'B' and AB can be calculated by differentiating *L*:

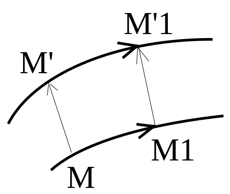
$$\partial L=\int_A^B \partial n ds+\int_A^B n\partial(ds)$$

Pay attention here to the difference between the quantities written as *dx* that represent an elementary displacement along a given trajectory and the quantities of the form ∂x that represent a small variation of the quantity *x* when we move from the path AB to the path A'B'.

The first term in ∂L corresponds to the effect of the index change ∂n when we move from point M on the final trajectory to a neighboring point M' in the modified trajectory such that **MM'** = ∂**M**. ∂n is given, as a function of the vector (**grad** n) considered at point M, by:

$$\partial n = \mathbf{grad} n . \partial \mathbf{M}$$

The second term in ∂L corresponds to the change in length of the trajectory. To evaluate the variation $\partial(ds)$, we will zoom on an elementary piece of the initial and deformed trajectories:



MM' = ∂**M**
$\mathbf{MM_1} = \mathbf{dM} = ds\, \mathbf{u}$
$\mathbf{M_1M'_1} = \partial \mathbf{M} + d(\partial \mathbf{M})$
$\mathbf{M'M'_1} = \mathbf{dM} + \partial(\mathbf{dM})$
Note that by writing $\mathbf{MM'} + \mathbf{M'M'_1} = \mathbf{MM_1} + \mathbf{M_1M'_1}$, we can identify d($\partial$**M**) and ∂(**dM**).

The length variation ∂(ds) can be written as:

$$\partial(ds) = \partial(\mathbf{dM.dM}^{1/2}) = \frac{\partial(\mathbf{dM}).\mathbf{dM}}{\mathbf{dM.dM}^{1/2}} = \partial(\mathbf{dM}).\mathbf{u} = d(\partial\mathbf{M}).\mathbf{u}$$

Lastly, the optical path variation ∂L can now be written as:

$$\partial L=\int_A^B \mathbf{grad} n.\partial\mathbf{M} ds+\int_A^B n\mathbf{u}.d(\partial\mathbf{M})$$

The second term can be integrated by parts:

$$\int_A^B n\mathbf{u}.d(\partial\mathbf{M})=[n\mathbf{u}.\partial\mathbf{M}]_A^B-\int_A^B d(n\mathbf{u}).\partial\mathbf{M}$$

We then get:

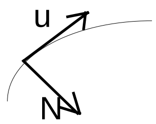
$$\partial L=n_B\mathbf{u_B}.\partial\mathbf{B}-n_A\mathbf{u_A}.\partial\mathbf{A}+\int_A^B(\mathbf{grad}\, n-\frac{d(n\,\mathbf{u})}{ds}).\partial\mathbf{M}ds$$

2) Application of Fermat’s principle. Equation for a light ray.

According to Fermat's principle, we know that the optical path followed by light between points A and B is the one that is stationary. In other terms, if we deform the trajectory followed by light keeping **A and B fixed,** the differential of the optical path must be zero. We can express this differential using the previous formula, where $\partial\mathbf{A}$ and $\partial\mathbf{B}$ are equal to zero since A and B are fixed. Since ∂L has to be zero whatever the deformation of the trajectory, i.e. for any $\partial\mathbf{M}$, the trajectory followed by light must verify:

$$\frac{d(n\,\mathbf{u})}{ds} = \mathbf{grad}n$$

This equation defines the trajectory of a light ray that is tangent to the unitary vector **u,** in an inhomogeneous medium characterized by the vector **grad** *n* in any point. We can also write this equation as a function of R, radius of curvature of the trajectory, **u** and **N** being unitary vectors respectively tangent and normal to the ray:



$\frac{d\mathbf{u}}{ds} = \frac{\mathbf{N}}{R}$ with $R > 0$ taking **N** pointing towards the center of curvature; thus the new equation for the light ray is:

$$\frac{dn}{ds}\mathbf{u} + \frac{n\mathbf{N}}{R} = \mathbf{grad}n$$

Taking the scalar product of the above equation with the vector **N**, we obtain the expression for the curvature of the trajectory:

$$\frac{1}{R} = \frac{1}{n}\,\mathbf{grad}n\,.\mathbf{N}$$

Since R is positive, the angle between **grad** *n* and **N** is always smaller than 90°. The concavity of the trajectory is always pointing in the same direction as the vector **grad** *n* (towards the areas with a higher index). It corresponds to what we had found with the simplified reasoning in the introduction.

Note: in the case when we deform the trajectory corresponding to the light ray by displacing its extremities A and B, the differential of the optical path is given by:

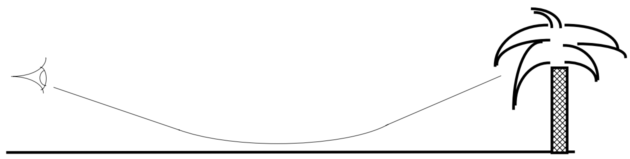
$$\partial L = n_B\mathbf{u_B}\,.\,\partial\mathbf{B} - n_A\mathbf{u_A}\,.\,\partial\mathbf{A}$$

We recover the result that we had shown in the previous chapter in the case of homogeneous media separated by refractive surfaces. The differential of the optical path along a light ray depends only on the displacement of its extremities.

III. Examples of applications

1) Mirages

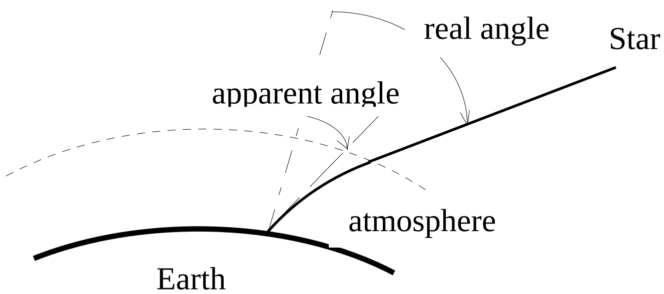
When the ground is very hot, a gradient in the air temperature appears, hotter near the ground and decreasing as you move up. Since the index of refraction of air (which is approximately equal to 1.0003) decreases when temperature increases ($\Delta n/\Delta T = -10^{-6}$ /°C). Air above the ground is thus an inhomogeneous medium with a vector **grad** *n* in the vertical direction pointing upwards. The rays coming from an object bend and reach the observer as if they had been reflected on a surface such as a water pool.



The mirage effect is also used to characterize optically the thermal properties of materials, by measuring the displacement of a laser beam as a function of heating.

2) Atmospheric refraction

The Earth's atmosphere shows a continuous variation of index with altitude related to the decrease of density: it changes from approximately 1.0003 near the ground up to 1 when you reach the vacuum. The rays coming from a star bend when they travel through the atmosphere so that the angle under which the star is seen from the ground is modified (by approximately 1 minute of arc).

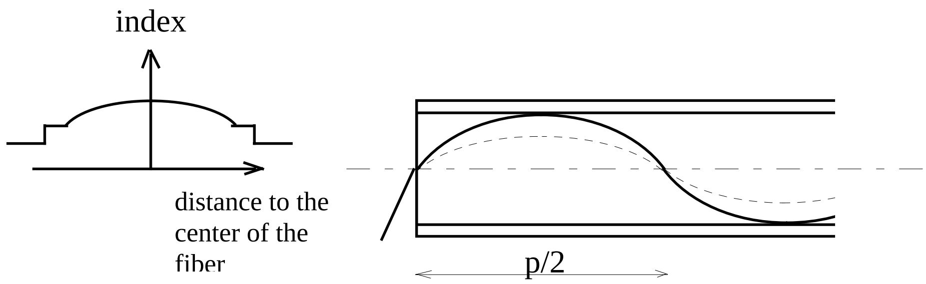


In particular, the sun can still be visible even after it has passed the theoretical horizon line. If you include the fact that the short wavelength rays (blue) are bent more than those corresponding to longer wavelengths (red) because the index of refraction is larger for shorter wavelengths, and also the fact that the eye's highest sensitivity is in the yellow-green, it explains the phenomenon of the “green ray”, visible at dusk just before the sun sets or in the morning just before it rises (only if the weather conditions are very good, and preferably in the mountains).

3) Gradient-index fibers and lenses

We have mentioned in the previous chapter the step-index fibers, made of two homogeneous media, a core with a higher index and a cladding with a lower index, where the propagation of light can be explained in terms of total internal reflection at the core-cladding interface (a more complete study of propagation in the waveguides in terms of modes of the electromagnetic field can be found in the electromagnetism course, the guided and coupled wave course or in reference books).

The following drawing illustrates the principle of gradient-index fibers, showing the index profile along a section of the fiber, and the trajectories of the rays:



We can show that for a well-chosen index gradient (of the type $n = n_0 - \alpha r^2$), all the trajectories are sinusoidal with the same period p. If we cut a section of such a fiber with a length shorter than p/2, we then get a gradient-index lens whose characteristics depend on the length chosen.