

## CHAPTER III

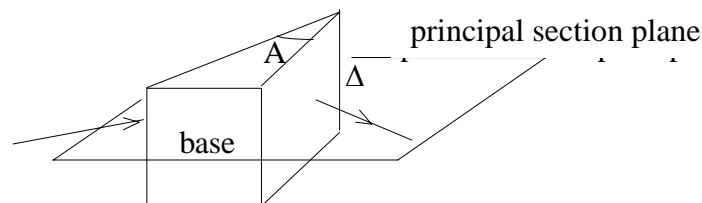
### APPLICATION OF SNELL-DESCARTES' LAWS TO THE STUDY OF PRISMS

A prism is a transparent medium limited by plane refractive surfaces that are not parallel. In this course, we will only consider prisms made of homogeneous and isotropic media. There are other types of prisms made of birefringent material (Wollaston, Rochon) that modify the polarization state of light ; those will be studied in the wave optics course.

We will see two types of prisms: dispersive prisms, used to separate the wavelengths of an optical spectrum, and the reflective prisms of which we have seen an example earlier with the right-angle prism (see chap. I). The first ones take advantage of the fact that refraction by index  $n$  depends on wavelength, the second ones use mostly the properties of reflection, especially what concerns the rotation and inversion of images.

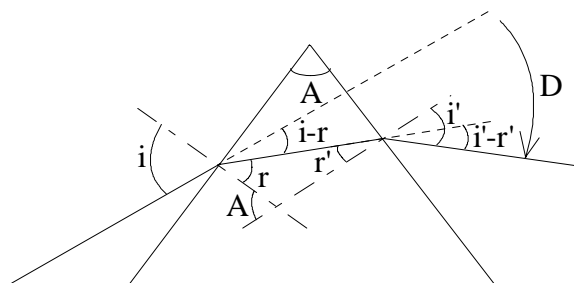
#### I. Dispersive prisms

A typical dispersive prism is made of two plane surfaces forming a triangular shape with edge  $\Delta$  and an angle  $A$  smaller than  $90^\circ$ .



$A$  is called the angle of the prism, the third side (vertical on the figure) is the base, and a plane orthogonal to the edge  $\Delta$  is a principal section plane of the prism. From now on, we will only consider rays traveling within a principal section plane.

#### 1) Formulas of the prism: deviation of an incident ray



Snell-Descartes' laws and the above figure allow us to write the following equations:

$$\begin{aligned}\sin i &= n \sin r \\ \sin i' &= n \sin r'\end{aligned}$$

$$r + r' = A$$

$$D = i - r + i' - r' = i + i' - A$$

*Note: in terms of signs, the angles are all considered positive in a configuration such as drawn above on the figure, which is the most common one for dispersive prisms. If we apply these equations to the general case, the case where  $r' < 0$  for example will correspond to an internal ray located below the normal to the output surface (case of a prism with a small angle  $A$ ). With this convention, the positive angles are defined differently at the input and at the output, but this has the advantage of keeping the symmetry due to the fact that light follows the same path in both directions.*

We can deduce from those equations the expression of  $D$  as a function of  $i$ :

$$D = i - A + \arcsin \left\{ n \sin \left[ A - \arcsin \left( \frac{\sin i}{n} \right) \right] \right\}$$

This expression is not very practical and it will be easier to determine the properties of the prism starting from the four equations above and studying the dependence of  $D$  with the angle  $A$ , with the incidence  $i$  and with the index  $n$  successively. This is what we are going to do in the following.

We can however note that the expression of  $D$  becomes very simple as soon as the angles  $A$  and  $i$  are sufficiently small so that we can linearize the laws of refraction. We then obtain:

$$D = i + i' - A = i + n \left( A - \frac{i}{n} \right) - A$$

$$D = (n - 1) A$$

## 2) Dependence of the deviation with the angle of the prism $A$

A simple reasoning allows us to figure out that  $D$  increases when  $A$  increases,  $i$  and  $n$  being fixed: if  $A$  increases,  $r'$  increases, thus  $i'$  increases. We can also obtain the expression of  $dD/dA$  by differentiating the four equations of the prism with  $i$  and  $n$  remaining fixed:

$$\cos i' di' = n \cos r' dr'$$

$$dr' = dA$$

$$dD = di' - dA$$

$$\frac{dD}{dA} = \frac{n \cos r'}{\cos i'} - 1 > 0$$

An important consequence of this equation is that the deviation  $D$ , which is zero when  $A$  is zero, has to be positive (with the sign of  $D$  defined as indicated on the figure). In other words, the deviation occurs towards the base of the prism.

## 3) Dependence on the angle of incidence $i$ . Conditions for the emergence of a ray.

We have considered so far the case when the incident ray comes out of the prism after two refractions. The first refraction is always possible (there is of course also a reflected ray) since we come from a medium of index 1 into a medium with a larger index of refraction  $n$ . On the contrary for the second surface glass/air, there can be total internal reflection that will keep the ray from emerging out of the prism. Let us find out the conditions for the ray to emerge.

### a) necessary condition of the angle of the prism A

The condition to avoid total internal reflection on the second surface is given by:

$$n \sin r' \leq 1 \quad \text{or also} \quad |r'| \leq i_\ell \quad \text{where} \quad \sin i_\ell = \frac{1}{n}$$

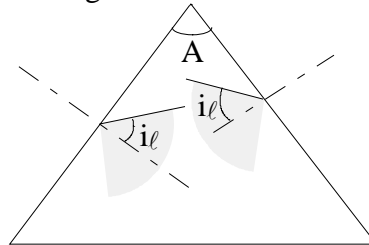
In particular  $r' = A - r$  has to be smaller than the limit angle  $i_\ell$  (the condition  $A - r \geq -i_\ell$  is automatically fulfilled since  $A > 0$ ). On the other hand, the largest refracted angle  $r$  is such that  $n \sin r = 1$ , so  $r$  is always smaller than  $i_\ell$ . We end up with the following condition on A:

$$A \leq r + i_\ell \leq 2i_\ell$$

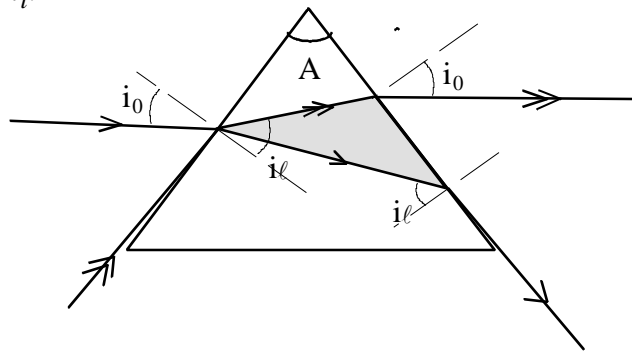
We can see that for a medium with an index of refraction  $n=1.5$  for example, all dispersive prisms must have an angle A smaller than  $2 \arcsin(1/n) = 84^\circ$ .

### b) conditions on the angle of incidence i

The easiest way to figure out those conditions is to draw a figure representing the ranges of accessible angles inside the prism, both for the incoming rays refracted with the angle  $r$ , and for the rays with angle  $r'$  that can emerge.



The rays that can emerge are those that are included in both gray areas (the above condition on the angle A for a non zero intersection reappears here). In the case of this figure where A is larger than  $i_\ell$ :



We can use these figures to determine the limit angles for the incident and emerging rays. Because of the reversibility of the light path, the range of accessible angles is the same at the input and at the output. In the case of the figure where  $A > i_\ell$ , we get:

$$i_0 \leq i \leq \pi/2$$

$$\text{where} \quad \sin i_0 = n \sin(A - i_\ell)$$

The case when A is smaller than  $i_\ell$  can be figured out as an exercise using a new figure to avoid errors in the signs of the angles.

### 4) Variation with the angle of incidence i. Minimum deviation.

We are interested here in the variation of the deviation  $D$  when we vary the angle of incidence  $i$  in the range where there is an emerging ray.

First we can note that, because of the reversibility of the light path, if a ray with an angle of incidence  $i$  emerges with an angle  $i'$ , a ray with angle of incidence  $i'$  will emerge of the prism with an angle  $i$ . If  $i \neq i'$ , it means that there are two different angles of incidence leading to the same value for the deviation  $D$ . Between those two values for the incidence angle, the variation of  $D$  is continuous so it will necessarily pass through an extremum (minimum or maximum), and this extremum can only occur when the incident and emerging angles are identical. In this case, the equations for the prism become:

$$r = \frac{A}{2} \quad i_m = \arcsin\left(n \sin \frac{A}{2}\right)$$

$$D_m = 2 \arcsin\left(n \sin \frac{A}{2}\right) - A$$

To show that this corresponds to a minimum and not to a maximum, we will go back to the equations that connect the different angles and we will differentiate them keeping  $A$  and  $n$  constant:

$$\begin{aligned} \cos i \, di &= n \cos r \, dr \\ \cos i' \, di' &= n \cos r' \, dr' \\ dr + dr' &= 0 \\ \frac{dD}{di} = \frac{di + di'}{di} &= 1 - \frac{\cos r'}{\cos r} \frac{\cos i}{\cos i'} \end{aligned}$$

We thus find that  $dD/di$  equals zero when  $i=i'=i_m$ . We also see that for  $i=\pi/2$ , this derivative equals 1: it is thus positive between  $i_m$  and  $\pi/2$ , and  $i_m$  thus corresponds to a minimum of  $D$ .

The experimental determination of this minimum deviation is easy, you just have to place the prism on a rotating platform with respect to the incident ray. From the measurement of  $D_m$ , we can then infer the value of the index of refraction of the prism using the following expression:

$$n = \frac{\sin(A + D_m)/2}{\sin A/2}$$

This commonly used method will be seen in the Optics lab Course.

## 5) Dispersion. Examples of dispersive prisms.

From the four equations of the prism, we can readily see that the deviation increases with the index of refraction: keeping  $i$  fixed, if  $n$  increases,  $r$  decreases,  $r'$  increases and  $i'$  increases. Since **in most optical materials, index of refraction decreases when wavelength increases** (so called 'normal' dispersion), we thus conclude that the prism will deviate blue light more than red light.

For example a prism made of BK7 glass (also called Crown glass) with an angle  $A=60^\circ$ , used with white light under an angle of incidence equal to  $40^\circ$ , will deviate blue light ( $\lambda=400\text{nm}$ ,  $n_{\text{BK7}}=1.530$ ) with an angle  $D_B = 41.8^\circ$  and red light ( $\lambda=650\text{nm}$ ,  $n=1.516$ ) with an angle  $D_R=40.2^\circ$ .

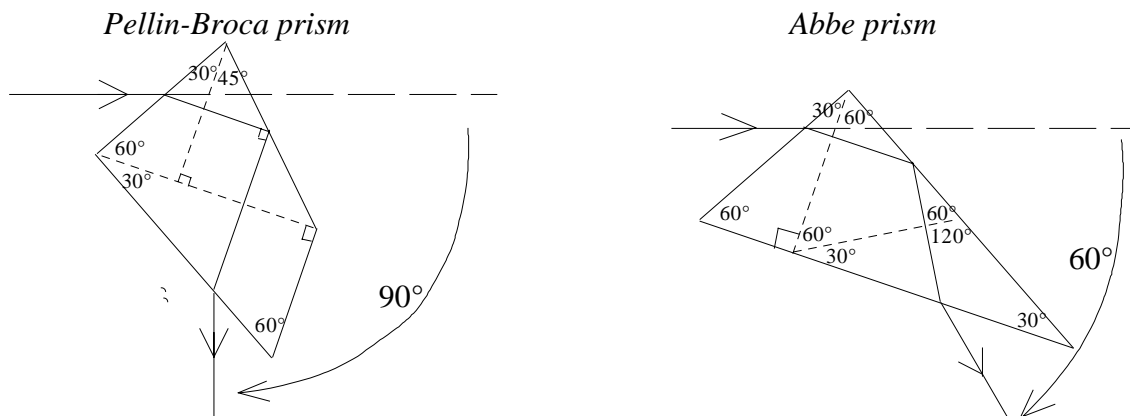
*Note: another commonly used material is Flint glass which has a larger dispersion but with a lower optical quality.*

We can be more precise and calculate the variation  $dD/dn$  by differentiating (as a function of  $n$  this time), the equations of the prism. For an angle of incidence  $i$  and an angle of the prism  $A$  fixed, the variation of  $n$  induces a variation of  $r$ ,  $r'$ ,  $i'$  and  $D$ :

$$\begin{aligned} 0 &= n \cos r \, dr + \sin r \, dn \\ \cos i' \, di' &= n \cos r' \, dr' + \sin r' \, dn \\ dr + dr' &= 0 \\ \frac{dD}{dn} &= \frac{di'}{dn} = \frac{\sin A}{\cos r \cdot \cos i'} \end{aligned}$$

Note finally that apart from the dispersive prism with a triangular shape that we have been describing so far, there are other useful shapes for dispersive prisms, with the interesting property that the deviation minimum is independent on the index of refraction thus on the wavelength. This value  $D_m$  equals  $90^\circ$  for the Pellin-Broca prism and  $60^\circ$  for the Abbe prism, and it corresponds in both cases to a refracted angle of  $30^\circ$ , identical at the input and the output of the prism. The difference with the previous dispersive prism comes from the fact that there is a total internal reflection inside the prism, which leads in particular to the fact that the angles  $i$  and  $i'$  have the same orientation.

To facilitate the understanding of the path of the rays, we have represented those two prisms as if there were made of three prisms whose limits are shown in dashed lines:

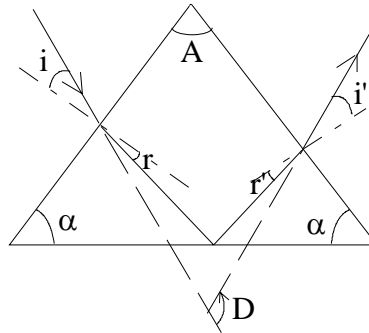


Let us take the example of the Pellin-Broca prism. If we send a parallel beam of white light on this prism and we observe in a fixed direction making an angle of  $90^\circ$  with respect to the incident beam, we will select in this direction a wavelength that will depend only on the orientation of the prism, and it will also be the deviation minimum for this wavelength.

## II. Reflecting prisms

We will now study a completely different kind of prisms, since we now want to use the properties of reflection (total internal or using reflective coatings) inside the prism and avoid dispersion. These prisms will be used either to deflect beams, or else to modify the orientation of images (single lens reflex camera, binoculars), sometimes both. They are thus related to combinations of mirrors. They have the advantage over mirrors of being monolithic, which means that they are aligned by construction and less sensitive to vibrations. Furthermore, in the case of total internal reflection prisms, the reflection coefficients are high over a large range of wavelengths.

The first condition to fulfill for a reflective prism to be effective is to avoid or reduce dispersion, which is usually connected to the refraction going in or coming out of the prism. Let us see how we can fulfill this condition in the case of a simple reflecting prism. We will consider an isosceles prism with an apex angle  $A$  that resembles much the dispersive prism considered earlier except for the light path being used:



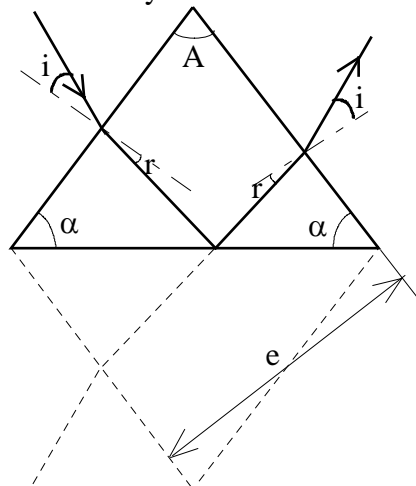
The symmetry of this isosceles prism leads to equal angles  $r$  and  $r'$  on one hand and  $i$  and  $i'$  on the other hand and we thus get a deviation:

$$D = 2i + A,$$

taking all angles positive on the above figure (be careful that the sign convention is different from the case of the dispersive prism).  $D$  is thus independent on the index of refraction thus on the wavelength. To indicate the difference with the dispersive prism, such a prism will be called an **achromatic** prism.

*Note: the internal reflection on the basis of the prism can be either a total internal reflection (which limits the range of useful incidence angles), or else the surface of the prism can be coated with a metallic layer.*

We can interpret the achromatic property of the above prism using a different method. Let us draw, by symmetry with respect to the plane of the mirror made of the basis of the prism, the path that is symmetric to the rays before reflection:

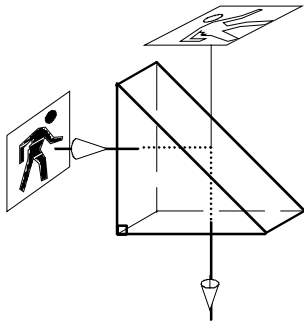


It all goes as if light was passing through a glass plate with parallel surfaces with a thickness  $e$  and an index  $n$ , that we can draw by symmetry of the prism with respect to the surface of the mirror. We find that the out coming angle is equal to the incoming angle. If we anticipate over the image formation in the paraxial approximation, for a plate with plane and parallel surfaces (see chapter VI), we shall see that the image of a small object through such a

prism will be translated by  $e(n-1)/n$ , in a direction normal to the output surface of the prism, with respect to the image of this object through the plane mirror.

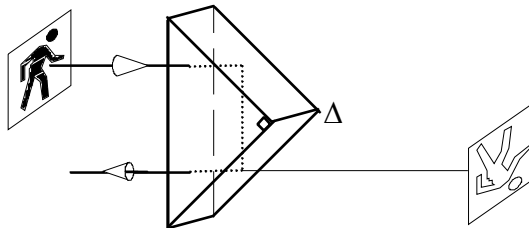
There are other shapes of achromatic prisms, some simple, some more complex, when we use as in the case of the isosceles prism the properties of symmetry to avoid dispersion. The figures below show a few of numerous achromatic prisms being used, for which we briefly mention the main properties.

*\* right angle prism*



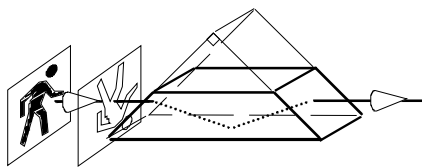
For an observer fixed with respect to the propagation of light, up and down are inverted, but left and right are unchanged.

*\* Porro prism*



This prism, identical to the previous one, gives in this configuration an image that can be superimposed with the object through a  $180^\circ$  rotation around the axis  $\Delta$

*\* Dove prism*



Once again it is the same right angle prism, where part of it has been cut off to reduce weight and size. It is used with a parallel beam.

**Review of the main properties of prisms**

\* for dispersive prisms, you should be able to write the four equations between the different angles in the prism. Using those, the dependence with the angle of the prism, the incidence angle and the index can be figured out by differentiating the four equations. The main consequences that should be remembered are:

- the deviation always occurs towards the basis of the prism;
- the conditions for a ray to emerge can be found using a drawing of the areas that are accessible inside the prism;
- there is a deviation minimum as a function of the angle of incidence, and it corresponds to a symmetrical situation when  $i=i'$ ;
- the deviation is larger for smaller wavelength.

\* A prism with a small apex angle  $A$  used for a small angle of incidence deflects the rays by a constant amount equal to  $(n-1)A$ .

\* there are reflective prisms that, thanks to symmetry properties, are not dispersive. They are often used to modify the orientation of images. A convenient way to find their properties consists in the determination of the mirrors+glass plate combination that is equivalent to the prism. In particular, in the same way as mirrors, the image will be identical to the object (sometimes through a rotation) if the number of reflections is even. It is a good idea to memorize one example of such a non-dispersive reflective prism.

Specific bibliography for this chapter:

*You can refer to the textbook « Optics » by E. Hecht (Addison-Wesley), in particular if you are looking for a larger list of reflective prisms (with perspective drawings and orientation of the images).*