# CHAPTER IV

# IMAGE FORMATION

# Looking for perfect stigmatism

We will now consider specific rays, those who come from an object and travel through an optical system in order to form an image.

The optical systems that we will consider in the following will almost always be made of **homogeneous and isotropic media**, separated by refractive surfaces (those are *dioptric systems*, such as the lens, the microscope, the Galilean telescope) and sometimes also reflective surfaces (*catadioptric systems*, including at least one mirror). They will mostly be *centered systems*, meaning that they will have an axis of symmetry (in this case, the ray traveling along the symmetry axis will not be deflected).

We shall see that the condition for an optical system to make a perfect image point of an object point (perfect stigmatism) is very constraining and that it is not fulfilled in most simple optical systems. There are however a few optical systems that are perfectly stigmatic for a specific couple of points object and image, and furthermore some of them remain approximately stigmatic in the close vicinity of these specific points. These systems are very useful when you want to observe very small objects with a large aperture, typically when you observe a star with a telescope or when you observe a cell in a microscope.

## I. Perfect stigmatism

### 1) Definition

An optical system is perfectly stigmatic for a pair of points A and A’ if all the rays originating at A intersect at A’ after passing through the optical system.

A' is then the perfect image of the object A formed by the optical system, and vice versa if we reverse the direction of propagation of light; A and A' are called conjugate points for this system.

*Reminder : we should not forget that in a real optical system, there will always be diffraction due to the finite diameter of the optical system. The perfect image of a point object will thus be a diffraction spot (an Airy function in the case of a circular diaphragm).*

### 2) Condition in terms of optical path

In terms of optical path, the perfect stigmatism translates into the fact that the optical path between points A and A’ is independent of the ray along which we are traveling:

$$L(\mathrm{AA'}) = cst$$

This can be proved by considering two rays close to each other originating at A: the stigmatism condition tells us that they will intersect at A', and Fermat's principle tells us that there is no change in optical path. Applying the same reasoning from one path to the next, we can then show that all the rays originating at A correspond to the same optical path from A to A'.

### 3) Condition in terms of wave surface

Thanks to the Malus-Dupin theorem, we can associate a wave surface to a beam of rays. For an object at a finite distance, the wave surface coming from the object is a portion of a sphere. If the object is at infinity, the wave surface is a plane. Similarly for the image, if all rays converge towards the same image point, the wave surface arriving at the image will be a portion of a sphere (a plane if the image is at infinity). Perfect stigmatism thus implies that the optical system transforms a spherical or plane wave surface into a spherical or plane wave surface.

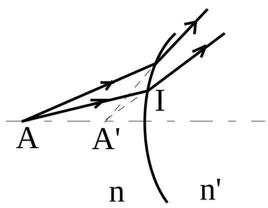
This property is often used to characterize the quality of an optical system. We send a plane or spherical wave at the input of an optical system, and we measure how the out coming wave surface differs from a perfect sphere, usually using an interferometric setup (Twyman-Green interferometer, Zygo interferometer).

### 4) Real and virtual objects and images

An optical system is limited by an input surface $S_1$ and an output surface $S_2$. We will usually represent light traveling from left to right.

An object is real if it is located to the left of the input surface $S_1$, virtual if it is on the right of $S_1$.
An image is real if it is located to the right of the output surface $S_2$, virtual otherwise.

Virtual objects and images no longer correspond to the intersection of light rays, but are located at the intersection of the virtual extension of these rays.

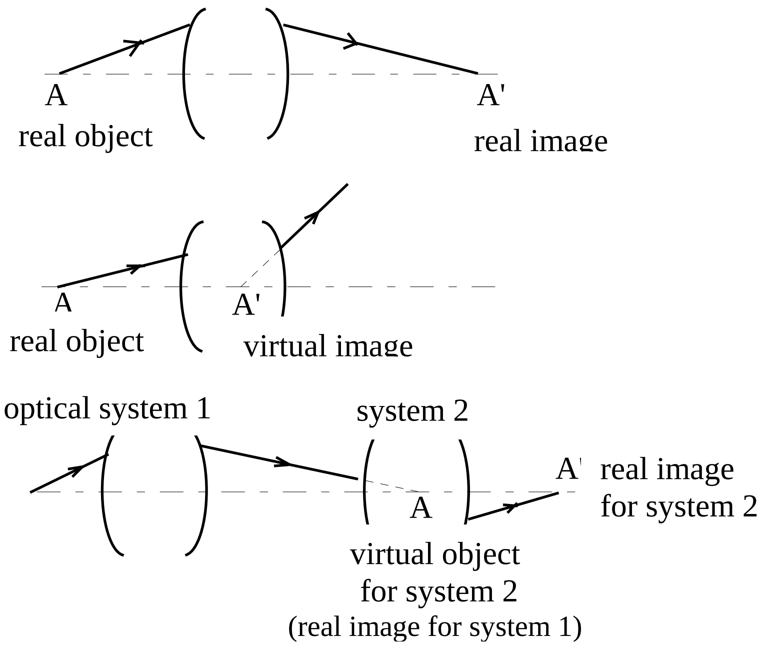



To calculate the optical path, virtual paths are counted with a negative sign. For example in the case of a surface between two media with indices n and n' (see figure above), the optical path from A to A' is given by:

$$L = n\overline{\mathrm{AI}} + n'\overline{\mathrm{IA'}}, \text{ with } \overline{\mathrm{IA'}}<0.$$

Note also that the index associated to the virtual path (IA') is the index of the image space *n'*, even though the emerging rays intersect geometrically to the left of the surface. In fact for the continuation of the path of this ray, it looks like there was only one medium of index n' where an object A' is located.

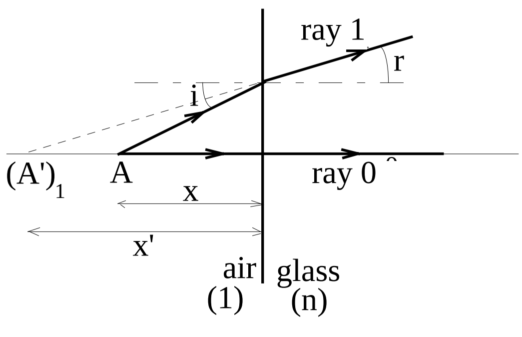
The following figures illustrate a few examples of different kinds of object/image pairs for centered optical systems (the ray, in mixed lines, traveling along the axis of symmetry of the system is not deflected).




## II Examples of optical systems that are not perfectly stigmatic

It is interesting to realize that most simple optical systems made of one or two surfaces, plane or spherical, do not fulfill the perfect stigmatism condition. We shall show this on two examples: the plane refractive surface and the spherical mirror.

Let us consider the case of a plane refractive surface between air with index 1 and glass with index 1.5. Let us consider an object at a finite distance A and look for its image A' through the refractive surface. The ray passing at A and normal to the surface is not deflected. Let us calculate the intersection of this ray with another ray originating at A and making an incidence angle i with the refractive surface.

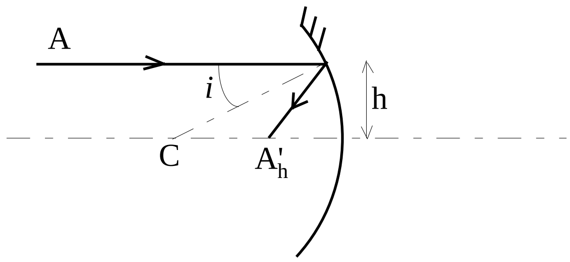



Applying Snell-Descartes' law for refraction, we can calculate the position *x*' of the intersection $(A')_1$ of rays 0 and 1:

$$x' = x\frac{\tan i}{\tan r} = nx\sqrt{\frac{1-\sin^2 i/n^2}{\cos^2 i}} = nx\sqrt{1+\tan^2 i(1-\frac{1}{n^2})}$$

We see that in general the position of the intersection with the optical axis will depend on the angle *i*, thus on the ray that was chosen: there is no perfect stigmatism. We notice that A' gets closer to the refractive surface as *i* decreases, and that the distance *x'* goes to *nx* when *i* goes to zero.

Let us consider now the case of a spherical mirror and of an object located at infinity.




We can calculate in a similar way as for the refractive surface the position of the intersection point $A'_h$ as a function of the height *h* of an incident ray originating at the object at infinity. We get:

$$\overline{CA'_h} = \overline{A'_h I} = \frac{R}{2\cos i} = \frac{R}{2\sqrt{1-h^2/R^2}}$$

As in the case of the refractive surface, the mirror is not perfect stigmatic, since the position of A' depends on the incident ray. We notice that A' moves away from the mirror when h decreases and it ends up in the middle of CS (focal point of the mirror).

The figures on the next page show the exact tracing (using Snell-Descartes' laws) of a large number of rays first for an object at a finite distance in front of a plane refractive surface, and second for an object at infinity on the axis of a spherical mirror.

In such situations, we never get a perfect image. The « imperfect » image (with aberrations) corresponds to the energy distribution in a plane perpendicular to the axis of the optical system. This image spot depends on the plane chosen for the observation.

Those two examples already show that aberrations on the image increase with the angle of incidence of the refractive surfaces or on the mirrors. We will thus often use the optical systems close to their optical axis.

## PLANE REFRACTIVE SURFACE

object

## SPHERICAL MIRROR

object at infinity on axis

## III. Perfectly stigmatic optical systems

Even though most optical systems are not perfectly stigmatic, there are a few useful cases where a specific optical system can fulfill the condition L=cst for one specific pair of conjugate points.

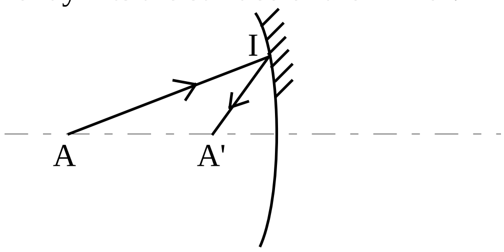
### 1) Optical systems made of mirrors (catoptric)

#### a) one mirror

For a mirror, the object and the image are in the same medium with index n so that the condition for perfect stigmatism *L=cst* is written as:

$$L=n(\overline{\mathrm{AI}}+\overline{\mathrm{IA'}})=cst$$

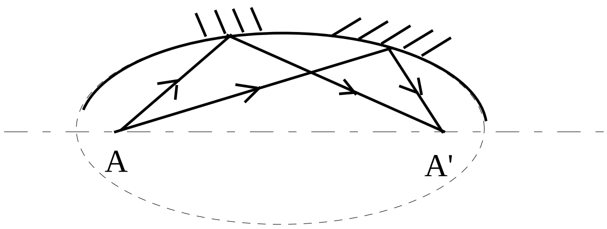
where I is the point where the ray hits the surface of the mirror.



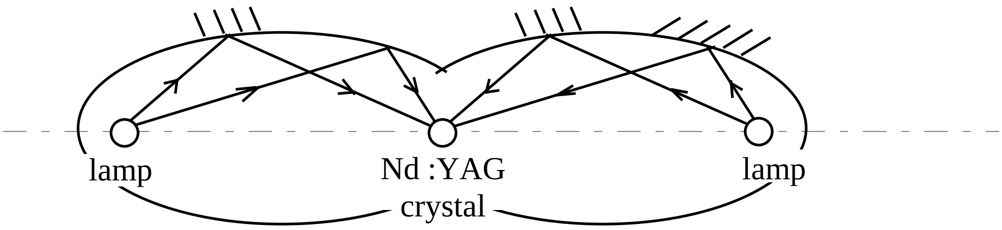

* The plane mirror fulfills this condition for any object point: the image is always symmetrically located with respect to the plane of the mirror ($\overline{\mathrm{IA'}} = -\overline{\mathrm{AI}}$, $L = 0$ for any ray). **The plane mirror is thus stigmatic for all objects.**

* The spherical mirror is perfectly stigmatic for its center of curvature ($\overline{\mathrm{AI}} = \overline{\mathrm{IA'}} = R$) and for all the points located on its surface ($\overline{\mathrm{AI}} = \overline{\mathrm{IA'}} = 0$). These points are their own image. This property can be used for example to increase the efficiency of a lamp: we increase the useful solid angle by placing the light bulb at the center of curvature of the mirror.

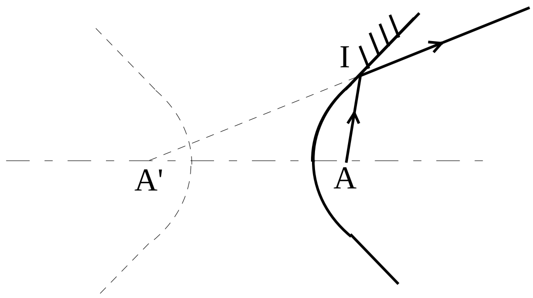
* For pairs of conjugate points of the same nature, both real or both virtual, $\overline{\mathrm{AI}}$ and $\overline{\mathrm{IA'}}$ have the same sign. The perfect stigmatism condition is fulfilled for an ellipsoid of revolution if A and A' are its foci. **The elliptical mirror is perfectly stigmatic for its foci.** (*note that we should not confuse the foci of the ellipse with the focal points of the optical system*).



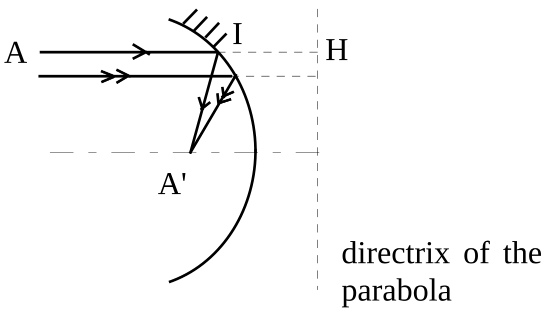

This kind of mirror is used for example to pump laser crystals such as Nd:YAG with a discharge lamp, the lamp being located at one of the foci of the ellipse and the laser crystal at the other (the section is elliptical, the system is invariant by translation in the perpendicular direction). If we want to increase the pump power by adding a second lamp, we then choose a double elliptic mirror: the two ellipses have one focus in common, where we place the laser crystal, and the two lamps are located at the two other foci:




* When the object and its image formed by the mirror are of a different nature (one real, one virtual), $\overline{AI}$ and $\overline{IA'}$ have opposite signs. The condition for stigmatism writes AI - IA' = cst (AI and IA' are the absolute values of $\overline{AI}$ and $\overline{IA'}$). The ensemble of such points I is a hyperboloid of revolution with foci A and A'. **The hyperbolic mirror is perfectly stigmatic for its foci.**




* When one of the two points is at infinity, for example the object A, we calculate the optical path between a plane wave surface arriving from the object, and the image A’. The plane wave surface can be chosen arbitrarily, we can thus choose it so that the optical path is zero. The stigmatic mirror will be the location of points at an equal distance between a point and a plane: it is a paraboloid of revolution with A’ as its focus. **The parabolic mirror is perfectly stigmatic for a pair infinity - focus.** (*in this case the focus of the paraboloid and the focal point of the optical system are identical*).



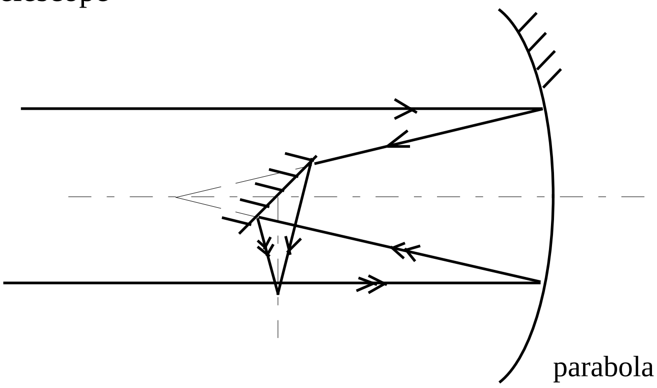

IA' = IH

We should point out that this kind of mirror rapidly gives low quality images when the object moves away from the ideal focus. It is used for example to collimate the light beam of a high intensity source (we place the source at A’ in order to get a parallel beam after reflection), to collect solar energy, and also very often in astronomy as the primary mirror.

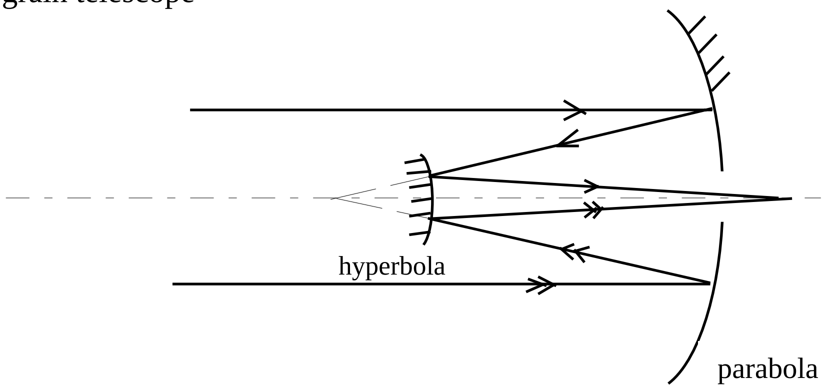
**b) Combination of mirrors - Reflective telescopes**

Even though conical shaped mirrors are much more difficult to make that plane or spherical surfaces, they are often used for mirrors in telescopes. The primary mirror is parabolic since we observe an object at infinity, the secondary mirror can be plane (Newtonian telescope), hyperbolic (Cassegrain telescope) or elliptical (Gregorian telescope). The corresponding configuration are drawn on the following figures:

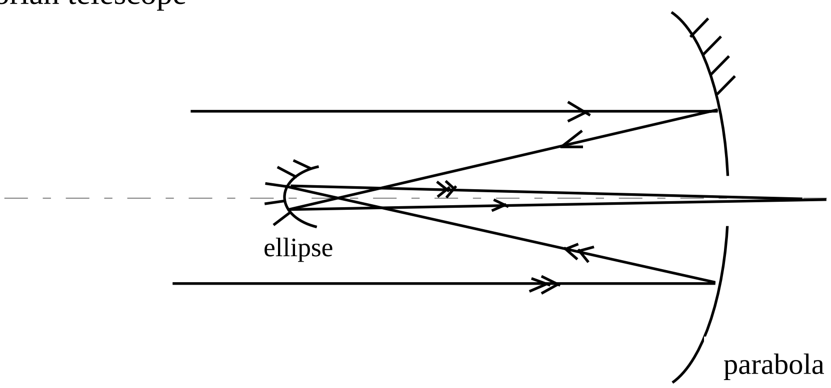
* Newtonian telescope




* Cassegrain telescope




* Gregorian telescope



## 2) Dioptric systems

### a) single refractive surface

As we saw earlier in this chapter, the plane refractive surface is not stigmatic, except for an object at infinity (then all angles of incidence are zero), which has an image at infinity.

For a surface between two media of indices *n* and *n'*, the stigmatism condition is written as:

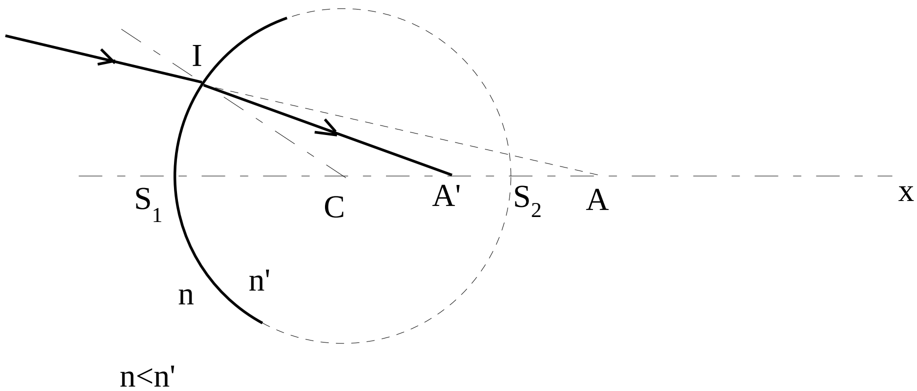
$$L = n\overline{\mathrm{AI}} + n'\overline{\mathrm{IA'}} = cst$$

This equation defines a family of curves, which are called *Descartes' ovals*.

A useful situation occurs when the constant is equal to zero. The condition is then given by $n\overline{\mathrm{AI}} + n'\overline{\mathrm{IA'}} = 0$, or else AI / A'I = cst. This is the equation of a sphere. The conjugate points A and A' that satisfy this property are called Young-Weierstrass' points. **The spherical refractive surface is stigmatic for the Young-Weierstrass' points.** Note that the spherical surface is obviously stigmatic for the points on its surface as well as for its center of curvature (object and image are identical).

According to the condition for stigmatism n $\overline{\mathrm{AI}}$ + n'$\overline{\mathrm{IA'}}$ = 0, $\overline{\mathrm{AI}}$ and $\overline{\mathrm{IA'}}$ have opposite signs, thus the object and its image are of a different nature, one real, one virtual.

In addition, since any ray passing through the center of curvature of the surface (thus along a diameter of the sphere) is not deflected, the two conjugate points are necessarily on the same diameter. By symmetry around C, the Young-Weierstrass' points will move on a circle depending on the part of the surface that is being used.




Let us call Ox the axis corresponding to one of the diameters of the sphere, and look for the distances $X = \overline{\mathrm{CA}}$ and $X' = \overline{\mathrm{CA'}}$ from the center of curvature to the object point A and image point A'. We clearly see that to fulfill the stigmatism condition for $S_1$ and also for $S_2$, the only solution is such that there is a change of sign, that is to say that the points A and A' that we are looking for are located on each side of point $S_2$. We can then write that the ratio of the distances from A and A' is the same in absolute value, equal to *n'/n*, for $S_1$ and $S_2$, which leads to:

$$\frac{\mathrm{S_1A}}{\mathrm{S_1A'}} = \frac{\mathrm{AS_2}}{\mathrm{S_2A'}} = \frac{n'}{n}$$

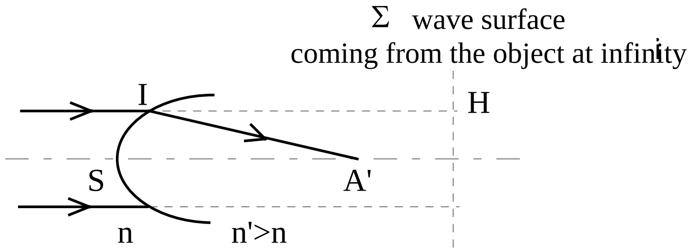
$$\frac{R+X}{R+X'}=\frac{R-X}{X'-R}=\frac{n'}{n}=\frac{2R}{2X'}=\frac{2X}{2R}$$

$$\text{thus:}\quad X=R\frac{n'}{n}\quad\text{and}\quad X'=R\frac{n}{n'}$$

Other special shapes of refractive surfaces can form a perfect image of an object at infinity. They must fulfill the condition:

$$n\overline{\mathrm{HI}}+n'\overline{\mathrm{IA'}}=cst \qquad \text{for any point I on the surface}$$

where $\overline{\mathrm{HI}}$ is the distance from a plane wave surface originating at the object up to the point I on the refractive surface. The plane wave surface can always be chosen such that the constant is zero. The figure below shows a situation where A' is real and $n'>n$.

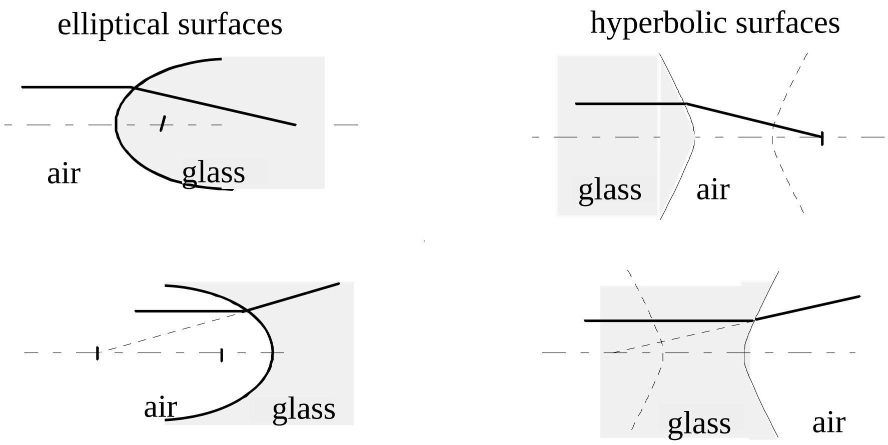



In this situation for example, the stigmatism condition can be written as:

$$\mathrm{IH}+e\mathrm{IA'}=0 \qquad \text{with } e>1$$

which defines an ellipsoid with a directrix Σ (eccentricity e>1), with one of its foci being A' (the one that is further away from S).

The four possibilities (A' real or virtual, $n'>n$ or $n'<n$) lead to the four stigmatic refractive surfaces below:

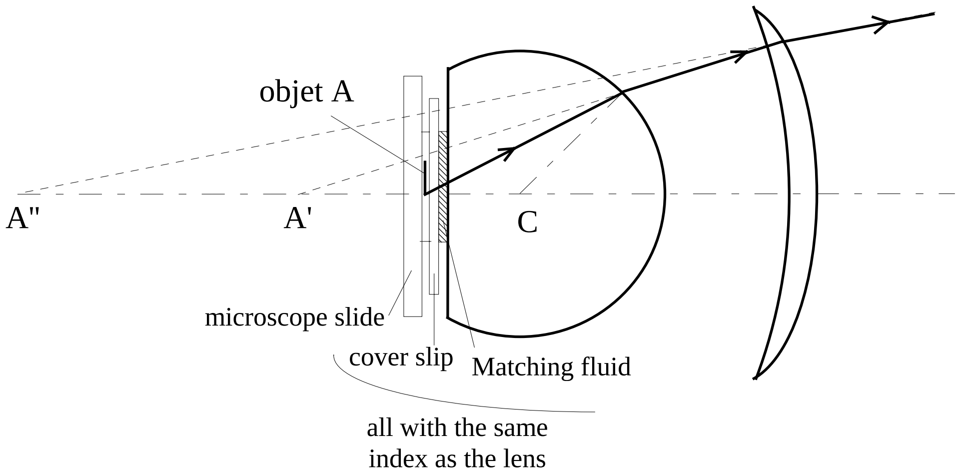



We will now see an example of use of these spherical refractive surfaces in condition of perfect stigmatism. Other combinations of refractive surfaces that form perfectly stigmatic lenses will be seen in chapter VII (§III.1).

### b) immersion microscope objective

With a microscope, it is important to be able to resolve details as small as possible. This requires both a good contrast and the best possible resolution limit for the optical system. However, even for a perfectly stigmatic optical system, diffraction limits the resolution to a size of the order of $\lambda/2n\sin\alpha$, where n is the index of the medium surrounding the object and $\alpha$ is the maximum angle between the optical axis and the rays entering the optical system. We thus want an objective with the largest numerical aperture $n\sin\alpha$, that can still make good images. It thus has to be perfectly stigmatic.

The object is placed at the one of the Young-Weierstrass points of a spherical glass-to-air refractive surface (point A on the previous figure). In order to include the object in the glass medium, we use a lens made of a sphere of glass partially truncated to leave access to point A and we immerse the object in a liquid with the same index as glass so as to reconstitute a continuous medium with index n.




At the output of the « half-sphere » lens, the aperture angle of the rays has decreased. If it is still too large, we can reduce it even more with a perfectly stigmatic meniscus as shown on the above figure. The first refractive surface is spherical with its center in A', so that the rays are not deflected but the image becomes virtual in the medium of index n (index of the meniscus). The second refractive surface is also spherical such that A' corresponds to its first Weierstrass' point. The final image is in A'', second Weierstrass' point of the last refractive surface. The out coming ray makes an even smaller angle with respect to the optical axis, and we can now use downstream optical elements that are not perfectly stigmatic.

A more detailed study of such a microscope objective, which can be done as an exercise, shows that the quality of the image is maintained if we move the object close to the optical axis, in a plane perpendicular to the axis containing the object point A. This property of the spherical refractive surface is called aplanatism. We will now describe it in a more general framework.

## IV. Approximate stigmatism in the vicinity of the perfectly stigmatic points

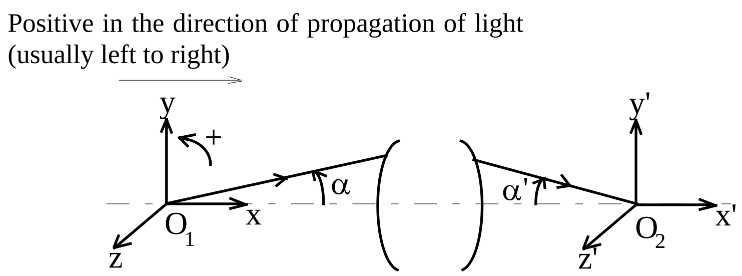
As a conclusion to the previous paragraph, we note that perfectly stigmatic systems often have complex shapes, and also that they are stigmatic only for one specific pair of conjugate points, which makes them less attractive to make the image of extended objects. However, we will have to distinguish among these systems which ones deteriorate the image more or less quickly as we move around those ideal conjugate points.

On the other hand, it is not always necessary to be in a situation of perfect stigmatism. Indeed, other phenomenon will deteriorate the quality of the image, such as diffraction that we mentioned already above in the case of the microscope, or such as the resolution of the detector (limited by the size of a pixel of the eye retina, or the grain of a sensitive emulsion, or of a CCD array). We must thus adjust the quality of the optical system to the whole experiment and define conditions for approximate stigmatism.

The approximate stigmatism that we will discuss in this chapter addresses the perfectly stigmatic systems that we have been considering before. Its goal is to find conditions for such a system to remain approximately stigmatic for conjugate points that are very close to the ideal ones, but with rays that still make a large angle with respect to the optical axis. We thus make an approximation on the size of the objects that we will observe, but not on the aperture of the optical systems. We will distinguish two cases, depending on whether the stigmatic condition is maintained in a plane perpendicular to the optical axis (aplanatism, Abbe’s condition) or along the optical axis (Herschel’s condition).

There is another way, more radical, to make any optical system able to be stigmatic for any object point: if we limit not only the size of the object but also the angle of incidence of the rays on the optical surfaces. It is the paraxial or gaussian approximation, which we will address at length in the next chapters because many optical systems can be used in that way.

### 1) Conventions for signs




We consider here a centered optical system with its axis called x in the object space and x’ in the image space. Positions in the object space are referred to a basis $O_1xyz$ and in the image space with respect to $O_2x'y'z'$. The positive sign for x and x' is defined in the direction of propagation of light (be careful in the case of system including mirrors). The two reference frames have the same orientation; their origins $O_1$ and $O_2$ are not necessarily conjugate points. The angles are measured algebraically between -90° and +90°, from the optical axis towards the ray (on the above figure $\alpha$ is positive and $\alpha'$ is negative).

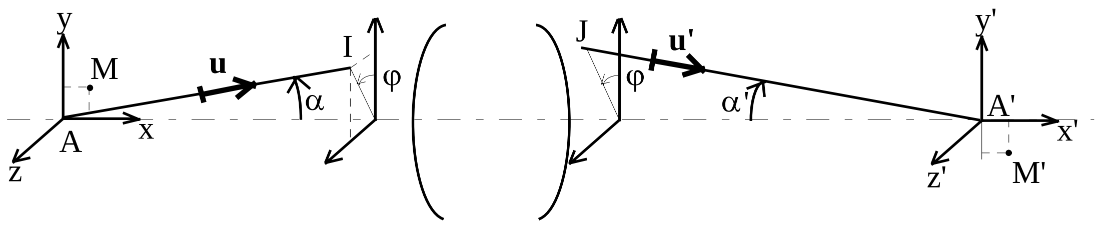
### 2) Approximate stigmatism in a small volume (for a centered optical system)

We consider an optical system that is perfectly stigmatic for a pair of conjugate points A and A'. We want to write the condition for two points M and M', respectively close to A and A', to satisfy the condition for stigmatism. An arbitrary ray originating at M and passing through a point I, will exit the system passing through M’ and the optical path (MIM') has to be independent of I.

Since M and M' are very close to A and A’, and since AIA’ is a path followed by light, the optical path (MIM') can be written as a function of the optical path (AIA') passing through the same point I, in the following way:

$$(\mathrm{MIM'}) = (\mathrm{AIA'}) + n'\mathbf{u'.A'M'} - n\mathbf{u.AM}$$

where $n$ and $n'$ are the indices of refraction of the object and image media, and **u** and **u'** are the unitary vectors along the incident ray originating in A and the out coming ray emerging through A’. Note that this expression is correct within the first order in AM and A'M'. This is where the approximation takes place, thus the *approximate* stigmatism.




Since the system is perfectly stigmatic for A and A', the optical path (AIA') is independent of I. The condition (MIM') independent of I can thus be written as:

$$n'\mathbf{u'.A'M'} - n\mathbf{u.AM} = cst \quad \textit{(independent on point I)}$$

Let us write this condition as a function of the coordinates of points M and M'. We choose as a reference frame Axyz so that M is in the Axy plane. By symmetry with respect to this plane, we show that M’ is also in this plane. We can also note that the symmetry with respect to the (AIA’) plane imposes that **u** and **u'** must be in the same plane including the optical axis, which makes an angle φ with respect to the plane of the figure.

*Note: this property uses the fact that the incident ray AI is in a plane that contains the optical axis. Note that a ray such as MI is NOT included in a plane including the axis.*

The coordinates of vectors **AM**, **A'M'**, **u** and **u'** can finally write:

$$\mathbf{AM}\begin{bmatrix} dx \\ y \\ 0 \end{bmatrix} \qquad \mathbf{A'M'}\begin{bmatrix} dx' \\ y' \\ 0 \end{bmatrix} \qquad \mathbf{u}\begin{bmatrix} \cos\alpha \\ \sin\alpha\cos\varphi \\ \sin\alpha\sin\varphi \end{bmatrix} \qquad \mathbf{u'}\begin{bmatrix} \cos\alpha' \\ \sin\alpha'\cos\varphi \\ \sin\alpha'\sin\varphi \end{bmatrix}$$

*Note: the quantities dx and y are both of the first order. This notation avoids later the confusion between x, the coordinate of an object point along the axis with respect to the origin of the reference frame and dx, a small displacement around such a point.*

The condition above turns into:

$$n'dx'\cos\alpha' - ndx\cos\alpha + (n'y'\sin\alpha' - ny\sin\alpha)\cos\varphi = \mathrm{cst} \quad \textit{(independent on } \alpha \textit{ and } \varphi\textit{)}$$

It must be verified for any ray, that is to say for any φ, thus:

$$n'y'\sin\alpha' = ny\sin\alpha$$

$$n'dx'\cos\alpha' - ndx\cos\alpha = \mathrm{cst}$$

The constant can be calculated for the ray along the axis ($\alpha = \alpha' = 0$), it is equal to $n'dx'-ndx$. We can then write the two equations in the following way:

$$n'y'\sin\alpha' = ny\sin\alpha \qquad \text{Abbe's condition}$$

$$n'dx'\sin^2\frac{\alpha'}{2} = ndx\sin^2\frac{\alpha}{2} \qquad \text{Herschel's condition}$$

### 3) Incompatibility of the Abbe's and Herschel's conditions in the general case

Let us choose a point M that is neither on the axis ($dx \neq 0$) nor in a plane perpendicular to the axis and containing A ($y \neq 0$). The Abbe and Herschel's conditions can be written as:

$$\frac{y'}{y} = \frac{n\sin\alpha}{n'\sin\alpha'}$$

$$\frac{dx'}{dx} = \frac{n\sin^2\alpha/2}{n'\sin^2\alpha'/2}$$

For a given point M and its image M', those two quantities must be constant for any ray, that is to say for any value of $\alpha$. Replacing in the first equation $\sin\alpha$ by $2.\sin\alpha/2.\cos\alpha/2$ (same thing for $\sin\alpha'$), we find that the ratio $\left|\cos(\alpha/2)/\cos(\alpha'/2)\right|$ must also be constant in absolute value. It is thus equal to its value when $\alpha=0$, in which case $\alpha'=0$ because of symmetry of revolution, so that:

$$\left|\frac{\cos\alpha/2}{\cos\alpha'/2}\right| = cst = \frac{\cos(\alpha/2=0)}{\cos(\alpha'/2=0)} = 1$$

$\alpha = \pm\alpha'$ (angles between -90° and 90°)

We thus see that if we allow for rays with a large angle with respect to the optical axis, the only stigmatic points around which the stigmatism can be approximately maintained in all directions are those for which the angular magnification is equal to 1 or –1. The transverse magnification is limited to the ratio of the indices of the object and image spaces (1 when they are identical), which is very restrictive. The plane mirror fulfills this condition for any object point, so does the spherical mirror for its center of curvature.

We shall see now that the condition is less stringent if we only want to maintain the stigmatism either in the perpendicular plane (observation of small objects), or along the axis (observation with a small depth of field).

*Note: in the case when we consider not only small objects (x and dy are small) but also small angles ($\alpha$ and $\alpha'$ small), the ratio of the cosines is equal to 1 in the first order even if $\alpha$ and $\alpha'$ are not equal. It is the paraxial approximation. In these conditions, Abbe's and Herschel's conditions give the relationships between the magnifications $g_y=y'/y$, $g_\alpha=\alpha'/\alpha$, $g_x=dx'/dx$, in the vicinity of points A and A':*

$$g_y g_\alpha = \frac{n}{n'} \qquad g_x\, g_\alpha^2 = \frac{n^2}{n'^2} \qquad g_x = \frac{n'}{n} g_y^2 \qquad g_y = g_x\, g_\alpha$$

### 4) Aplanatism: Abbe's condition

We go back to the previous equations in the case when M is in the same plane orthogonal to the optical axis as A ($dx$=0). The Herschel's condition, which must be true for any incident ray, i.e. for any $\alpha$, imposes that $dx'$=0: the image must then be in the same orthogonal plane as A'. Let us show that this condition is always fulfilled within the first order in y:

> *Since we consider only first order terms, the most general expression for dx' is dx'=ay+b. We know that if M=A, then M'=A': b is thus equal to zero. In addition the symmetry of revolution of the system imposes that if $M_1$ is symmetric to M with respect to the axis (i.e. $y_1 = -y$), its image $M'_1$ is symmetric to M' with respect to the axis: $dx'_1 = dx'$, $y'_1 = -y'$. This implies that dx' must be an even function of y, so that a=0.*

The condition for approximate stigmatism in the orthogonal plane A$yz$ is Abbe's condition: a system that fulfills it will be called **aplanatic**.

The conditions for aplanatism of an optical system are thus:

(*a*) the optical system is perfectly stigmatic for a pair of conjugate points A and A';
(*b*) Abbe's condition is fulfilled for any ray originating at A.

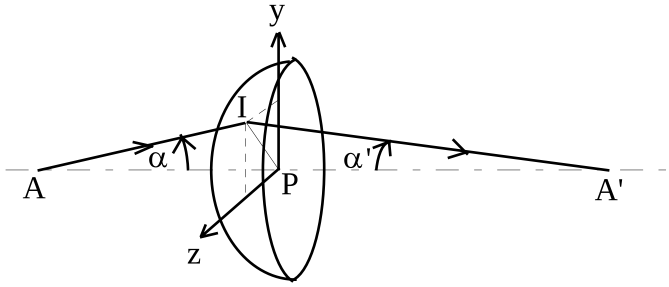
An aplanatic system will give good images of small objects perpendicular to the optical axis, even for rays that make a large angle with respect to the axis.

This condition for aplanatism or Abbe's sine condition will be very useful in the study of aberrations of optical systems. In terms of aberrations, the 2 above conditions take the following form:

(*a*) the optical system is corrected from spherical aberration for A and A' (it is the only aberration present in a centered system for an object on axis);
(*b*) the optical system is corrected from coma.

The fact that for an object B with a small size y in the same plane as A, the quantity $dx'$ varies as $y^2$ is an aberration called field curvature.

Let us see if these aplanatism conditions are fulfilled in the case of the simple systems that we listed in the previous paragraph as being perfectly stigmatic for conjugate points at a finite distance (spherical refractive surface, elliptical and hyperbolic mirrors). Since those systems have only one surface, refractive or reflective, we completely define the rays by the positions of the impact point I on the surface.



P being the projection of I on the axis of the system, we can write:

$$\sin\alpha = \frac{\overline{PI}}{\overline{AI}} \qquad \sin\alpha' = \frac{\overline{PI}}{\overline{A'I}}$$

Abbe's sine condition can be written as:

$$\frac{\sin\alpha'}{\sin\alpha} = \frac{ny}{n'y'} = \text{cst} = \frac{\overline{AI}}{\overline{A'I}}$$

This can only be true if I is on a sphere.

As a conclusion, the spherical refractive surface is aplanatic for its center and for the Young-Weierstrass' points. So is the spherical mirror for its center of curvature. On the contrary, the elliptical and hyperbolic mirrors are not aplanatic.

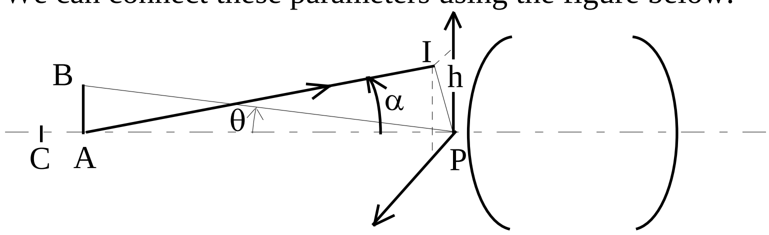
### 5) Herschel's condition

We now choose a point M in the vicinity of A along the axis (y=0). By symmetry, its image will also be on the axis, and Abbe's condition is automatically true. The approximate stigmatism is thus fulfilled if Herschel's condition is fulfilled for any ray with an angle $\alpha$ originating at A.

A perfectly stigmatic system for a pair of points A and A', that fulfills Herschel's condition will give good images of point objects even if they are displaced a little bit along the axis, no matter how large the aperture is.

### 6) Abbe's condition for object or image at infinity

When the object, for example, is very far away, the convenient parameters are not *y* and $\alpha$ anymore, but the angle $\theta$ under which the object AB is seen and the height h of the ray originating at A. We can connect these parameters using the figure below:




The angles $\alpha$ and $\theta$ are small since AP goes to infinity:

$$\alpha = \frac{h}{AP} \qquad \theta = -\frac{y}{AP}$$

Abbe's condition is thus given by:

$$n'y'\sin\alpha' = -nh\theta$$

We can write in the same way Abbe's condition for an image at infinity by replacing $n'y'\sin\alpha'$ with $-n'h'\theta'$.

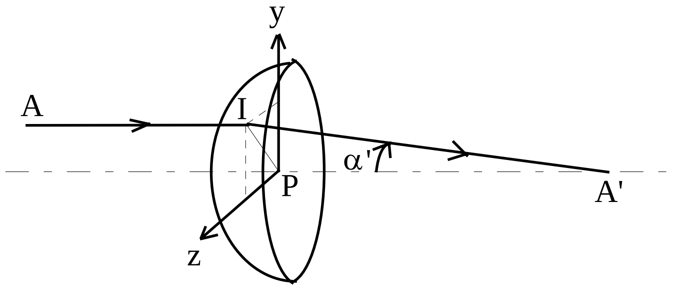
For Herschel's condition, we will characterize the position of a point far away along the axis by its proximity $\xi_A$= 1/PA for point A, and $\xi_C$ = 1/PC for point C close to A on the axis. We can connect $d\xi = \xi_C - \xi_A$ to $dx$=AC, knowing that C and A are close:

$$dx = -\frac{d\xi}{\xi_A{}^2}$$

In addition we have $\alpha = -h\xi_A$, which turns Herschel's condition into:

$$n' dx' \sin^2 \frac{\alpha'}{2} = -\frac{1}{4} n h^2 d\xi$$

Let us see what the aplanatism condition implies for a single surface (refractive or reflective). In particular, we want to know whether the parabolic mirror, which is stigmatic for an object at infinity on the axis, is aplanatic. For a system made of a single surface, the incident and emerging rays cross in one point I on the surface:




We can write Abbe's condition for an object at infinity in the form:

$$\frac{h}{\sin\alpha'} = -\frac{n'y'}{n\theta} = cst$$

here: $$\frac{\overline{PI}}{\sin\alpha'} = \text{cst} = \overline{A'I}$$

Abbe's condition implies thus that I moves on a sphere with center A' when the ray originating at A varies. The parabolic mirror is thus not aplanatic, neither are all the non-spherical refractive surfaces (seen in §III-2a).

**Conclusion:**

In this chapter, we have seen that there are a few optical systems that are perfectly stigmatic : plane mirror for all points, spherical mirror for its center of curvature, elliptical and hyperbolic mirrors for their geometrical foci, parabolic mirror for an object at infinity on axis, spherical refractive surface for its center of curvature and the Young-Weierstrass' points. Among all these systems, only those that are spherical or plane are aplanatic.

There are of course other optical systems, more complex, that are stigmatic even for large aperture angles (Ritchey-Chrétien telescope, wide angle camera lenses). They are made of the association of systems whose aberrations compensate one another (we have seen for example that the aberration on axis of a concave mirror has the opposite sign than the aberration of a plane refractive surface). This compensation is designed depending on the type of images that we want to make (wide field but small aperture, or small field with a large aperture…). The aplanatism condition will reappear in the study of aberrations. These systems will be studied in the Optical Design course.