CHAPTER VI

Study of simple elements: THE SINGLE REFRACTIVE SURFACE

I. The plane refractive surface.

1) Elementary properties. Stigmatism.

A plane refractive surface separates two homogeneous media with different indices. We will study here only the refracted part of the light on this surface (otherwise the surface will be studied as a plane mirror). Let us recall than Snell-Descartes' law states that the refracted ray lays in the plane of incidence and that the angle i_2 between that ray and the normal is given by: $n_2 \sin i_2 = n_1 \sin i_1$.

We have seen in chapter I, section IV, the two methods that allow us to construct the refracted ray in the case of an arbitrary incidence angle (even a large one).

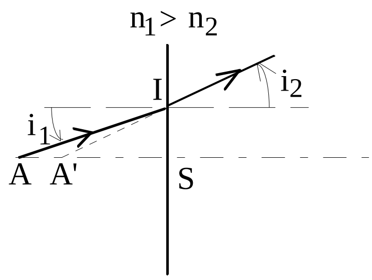
We have also seen in chapter IV, section II, that the plane refractive surface is not stigmatic, except for points on its surface or for an object at infinity. In this last situation, all rays have the same angle of incidence, so that all refracted rays are parallel: the image is at infinity, so the system is afocal.

2) The plane refractive surface in the paraxial approximation

We can recover the properties of the plane surface in the paraxial approximation either using the calculations done for the spherical surface (chap., section I) with a radius of curvature going to infinity or else going back to the calculation of the position of the image as a function of the angle of incidence (chap. II, section II). Here, we will rather use a geometrical construction where we will linearize the law of refraction in the following way:

$$n_1 i_1 = n_2 i_2$$

a) position of the image



$n_1 i_1 = n_2 i_2$ can be written here:

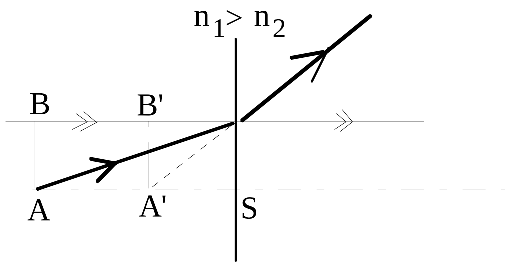
$$n_1 \frac{\overline{SI}}{\overline{SA}} = n_2 \frac{\overline{SI}}{\overline{SA'}}$$

From this we get the position of the image:

$$\frac{\overline{SA'}}{\overline{SA}} = \frac{n_2}{n_1}$$

b) Image of an object. Magnifications.

In the case of a plane refractive surface, which is afocal, the only specific rays that can be used for constructions are the rays parallel to the axis which, as we know, are not deflected. This is not enough to construct an image, so we need to use the previous relationship giving the position of the image.



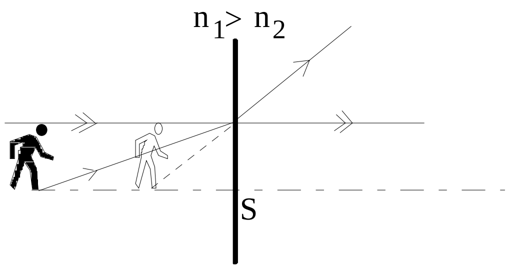
However it is easy to see from the construction that the transverse magnification is equal to 1 for all points: the ray parallel to the axis passing through B is not deflected, so B’ is on that same ray, at the same distance from the axis as B.

The angular magnification is directly inferred from the law of refraction since the angles with respect to the axis are equal to the angle with respect to the normal: $g_\alpha = n_1/n_2$. Note that the Lagrange-Helmholtz theorem $ny\alpha = n'y'\alpha'$ is true for the plane refractive surface.

The axial (or lateral) magnification is obtained by differentiating the formula giving the position of the image:

$$\frac{x'}{x} = \frac{\overline{SA'}}{\overline{SA}} = \frac{n_2}{n_1} \qquad \text{so that: } \frac{dx'}{dx} = \frac{x'}{x} = \frac{n_2}{n_1}$$

The axial magnification is smaller than 1 in the case of our figure ($n_1 > n_2$), so that the proportions of our pedestrian are not conserved:



Let us summarize the expressions of all the magnifications:

$$g_y = 1 \qquad g_\alpha = \frac{n_1}{n_2} \qquad g_x = \frac{n_2}{n_1}$$

Note 1: we get back the relationship between magnifications $g_y = g_x . g_\alpha$ *that, together with the Lagrange-Helmholtz’s theorem, gives all the other magnifications knowing the transverse one.*

Note 2: Even though the transverse magnification is equal to 1, a gold fish observed through the surface of a fish bowl will look larger than it really is. This is due to the fact that the image is closer to the surface than the object (for $n_1 > n_2$*), so it is also closer to the eye of* *the observer. We will talk again about those effects due to the fact that the eye is in itself an optical system.*

II. The plane and parallel plate

A plane and parallel plate is a homogeneous and transparent medium limited by two plane refractive surfaces that are parallel. The plate is characterized by its thickness e and its index n. We will consider it immersed in a medium with index n_1, identical on both sides of the plate.

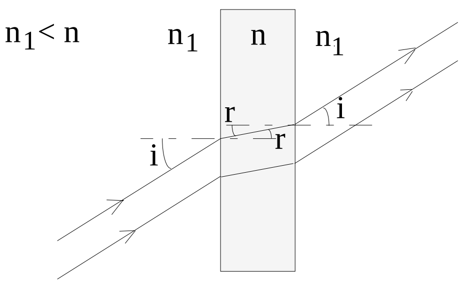
1) properties of plane and parallel plates

A plane and parallel plate refracts an incident parallel beam into a parallel emerging beam, for any incidence angle of the rays.

We can prove this for example in terms of images: the object being at infinity, the plane refractive surfaces make an image at infinity. In fact this property is true even if the two surfaces are not parallel but only plane (we are talking about a prism), as well as in the case when the indices on both sides of the plate are not identical.

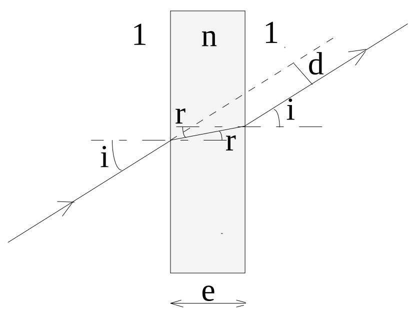
When the indices on both sides of the plate are identical, a plane and parallel plate does not deflect the beam: the emerging rays are parallel to the incident rays.

It is easy to prove on a figure using the symmetry by inversion of the direction of propagation:



2) Lateral displacement of the beam

We are interested in the lateral displacement *d* induced by a plane and parallel plate on an incident ray with an incidence angle *i*. We will consider here a plate with thickness *e* an index *n*, immersed in air (n_1=n_2=1).



d can be written, as a function of the angles *i* and *r*, as:

$$d = \frac{e\sin(i-r)}{\cos r}$$

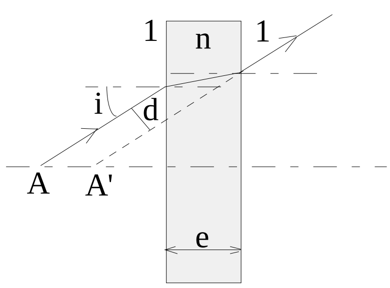
We can see the dependence with the index of refraction if we replace *r* by arcsin(*i*/*n*). Note that this displacement is larger if the plate is thick and it varies from 0 to e when the angle of incidence varies from 0 to 90°. For example for a glass plate with thickness 1 cm and index n=1.5, under the incidence i=45°, the displacement *d* is equal to 0.33cm.

This property of plane and parallel plates is very useful to translate a beam by a small amount: the absence of deflection is guaranteed by the parallelism of the plate, which can be tested very precisely using interferometric methods. On the contrary, to perform the same translation using plane mirrors, you will need a mechanical translation of very high quality, which is much harder to get.

3) Image through a plane and parallel plate

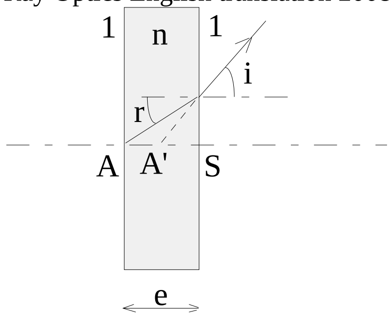
For the same reason as the single plane refractive surface, the plate is not perfectly stigmatic except for an object at infinity. For an object at a finite distance, the stigmatism is true only in the linear approximation.

Before we calculate the position of the image of a point A, note that when the extreme media are identical, this position does not depend on the distance from the object to the plate. This can be demonstrated on a construction, when we trace one ray at normal incidence and one ray at an incidence angle *i*:



If we translate the plate while considering the same angle of incidence *i*, the position of A' does not change since it depends only on the distance *d* and on the angle *i*. In fact, since we are in the paraxial approximation, we know that the position of A’ should not depend on the angle of incidence.

To calculate the distance AA', we can thus consider the case when the first surface of the plate is right on the object A.



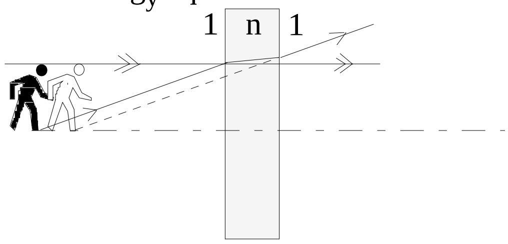
The image through the first surface is A itself. A' is thus the image of A through the second glass/air surface located at a distance *e* from A, so if we use the formula we demonstrated about the single plane refractive surface:

$$\frac{\overline{\mathrm{SA'}}}{\overline{\mathrm{SA}}}=\frac{1}{n}=\frac{-e+\overline{\mathrm{AA'}}}{-e},$$

We get the distance between object and image:

$$\overline{\mathrm{AA'}}=e\,\frac{n-1}{n}$$

Finally, in terms of transverse magnification, the plate has, for the same reason as the single plane surface, a magnification g_y equal to 1:

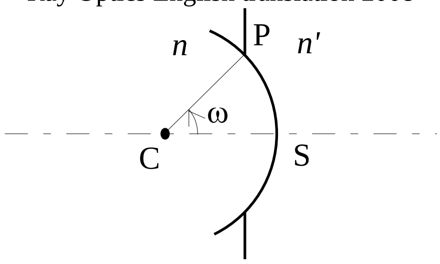


Note that for identical media on both sides of the plate, the axial magnification *dx*'/*dx* is also equal to 1, which is different from the case of a single plane refractive surface.

III. The spherical refractive surface

1) Definitions

A spherical refractive surface is a portion of a sphere that separates two homogeneous and isotropic media with indices of refraction *n* and *n*'. The useful part of the sphere in the paraxial approximation is limited by a plane P. We define the center C of the surface (center of the sphere), the vertex S (intersection with the optical axis), the radius of curvature $R=\overline{\mathrm{SC}}$, positive in the direction of propagation of light, and the aperture angle ω of the portion of sphere. The axis SC is the principal axis of the refractive surface, all the other axes passing through C are secondary axes, a plane containing the axis is a principal section plane and a plane orthogonal to the axis is a front plane.



2) Stigmatism

Let us review the results that have been demonstrated in the previous chapters. The spherical refractive surface is perfectly stigmatic and aplanetic for the points on its surface, for its center of curvature (which are their own images) and for the Young-Weierstrass points defined by:

$$L = cst = n\overline{AI} + n'\overline{IA'} = 0 \qquad \forall \text{I on the spherical surface },$$

from which we can deduce the positions of these points and the corresponding transverse magnification:

$$\overline{CA} = R\frac{n'}{n} \qquad \overline{CA'} = R\frac{n}{n'} \qquad g_y = \left(\frac{n}{n'}\right)^2$$

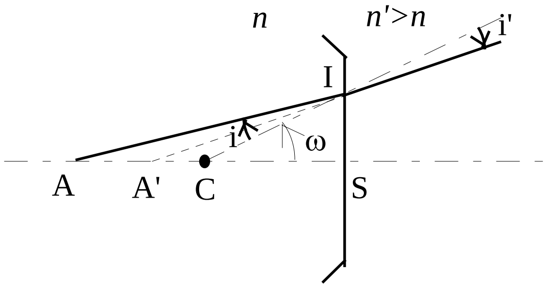
3) The spherical refractive surface in the paraxial approximation

We have seen in the beginning of chapter V the paraxial approximation applied to the spherical refractive surface, which led in particular to the gaussian formula for the position of the image. We can also easily see that H=H'=S and N=N'=C. Thanks to the general demonstrations in the paraxial approximation, we know that in that approximation the spherical surface is stigmatic for all the points on axis and close to the axis, and that the image of a front plane is a front plane.

Rather than applying this general formalism, somewhat complicated for the simple case of a single refractive surface, we will demonstrate in what follows that all the properties of the spherical refractive surface (position of the image, magnifications) from geometrical constructions. We recall here that in the linear approximation, we can replace the spherical surface by their tangent plane: we will thus represent the spherical surface by a plane with an indication on its edges of the direction of its curvature. The figures will be represented with different scales along x (the axis) and y (transverse direction) so that small angles are still visible.

a) gaussian formula with origin at the vertex

We want to connect the position of the object A and of its image A' to the position of the vertex S of the spherical surface. In order to do that, we draw the following figure:



$$\omega - i = -\frac{\overline{SI}}{\overline{SA}} \qquad \omega - i' = -\frac{\overline{SI}}{\overline{SA'}} \qquad \omega = -\frac{\overline{SI}}{\overline{SC}}$$

The linearized law of refraction $n\,i = n'\,i'$ turns into:

$$n\,(-\frac{\overline{SI}}{\overline{SC}} + \frac{\overline{SI}}{\overline{SA}}) = n'\,(-\frac{\overline{SI}}{\overline{SC}} + \frac{\overline{SI}}{\overline{SA'}})$$

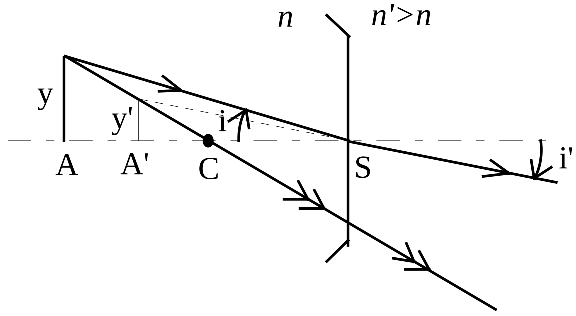
for which we get the gaussian formula with origin at the vertex S:

$$\frac{n'}{\overline{SA'}} = \frac{n}{\overline{SA}} + \frac{n'-n}{\overline{SC}}$$

It is possible to write a gaussian formula with origin at the center C. This can be done as an exercise. However these two formulae look alike without being exactly the same so the risk of confusion is high. It is thus recommended to memorize only the gaussian formula with origin at S, which is the easiest one to verify graphically.

b) magnifications

To find the transverse magnification, we draw the following figure:



The simplest expression for g_y is obtained when A and A' are referred to the position of the center of the spherical surface using the ray passing through C which is not deflected (ray with the double arrow):

$$g_y = \frac{\overline{CA'}}{\overline{CA}}$$

We can also on the same figure find the transverse magnification with the origin at S thanks to the ray passing through S which comes out with an angle *i'* connected to the incidence angle *i* by:

$$n\,i = n\frac{y}{\overline{SA}} = n'\,i' = n'\frac{y'}{\overline{SA'}}$$

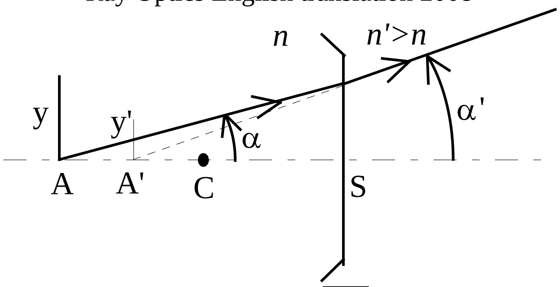
$$g_y = \frac{n}{n'}\frac{\overline{SA'}}{\overline{SA}}$$

From those two expressions of gy, we find that the quantity $n\overline{CA}/\overline{SA}$ is conserved in the image space:

$$n\frac{\overline{CA}}{\overline{SA}} = n'\frac{\overline{CA'}}{\overline{SA'}}$$

This relationship is in fact equivalent to the Lagrange-Helmholtz's theorem $ny\alpha = n'\,y'\,\alpha'$.

We can deduce the angular magnification from the transverse magnification using Lagrange-Helmholtz's theorem, or directly from the following figure:



$$g_\alpha = \frac{\alpha'}{\alpha} = \frac{\overline{SA'}}{\overline{SA}}$$

As for the axial magnification, we get it for example by differentiation of the gaussian formula with origin at S, writing $x = \overline{SA}$ and $x' = \overline{SA'}$:

$$g_x = \frac{dx'}{dx} = \frac{d(\overline{SA'})}{d(\overline{SA})} = \frac{n}{n'}\frac{x'^2}{x^2}$$

We can summarize the expressions for the 3 magnifications:

$$g_y = \frac{n}{n'}\frac{x'}{x} \qquad g_x = \frac{n}{n'}\frac{x'^2}{x^2} \qquad g_\alpha = \frac{x}{x'}$$

We recover the general relationship between magnifications for any centered system $g_y = g_x . g_\alpha$, which allows, together with the Lagrange-Helmholtz's theorem, to find all the magnifications from the specification of one of them, usually the transverse one.

c) Focal points. Focal lengths.

The principal planes are both located in S. We can get the position of the focal points using the previous gaussian formula:

- the first focal point has its image at infinity:

$$\overline{SF} = \overline{SC}\frac{n}{n-n'} = f$$

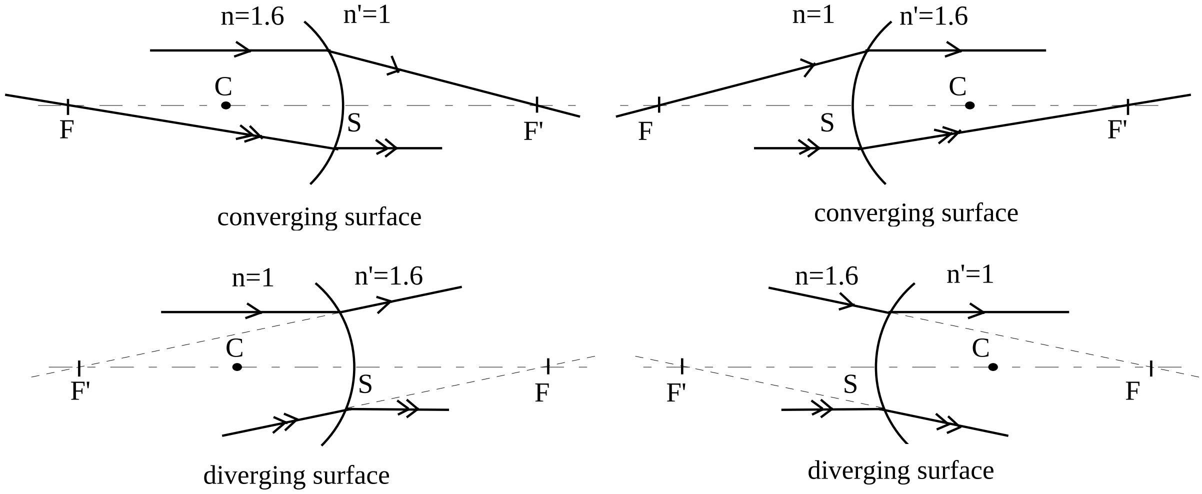
- the second focal point is the image of an object at infinity:

$$\overline{SF'} = \overline{SC}\frac{n'}{n'-n} = f'$$

The power P is equal to:

$$P = \frac{n'}{f'} = -\frac{n}{f} = \frac{n'-n}{R}$$

We recover that, as for any dioptric system, the focal lengths have opposite signs and are proportional to the indices of the extreme media.
Here in addition, the focal points are located on opposite sides of S. When the spherical surface is converging (P>0), the two focal points are real. When the spherical surface is diverging, the two focal points are virtual. Here are four examples:

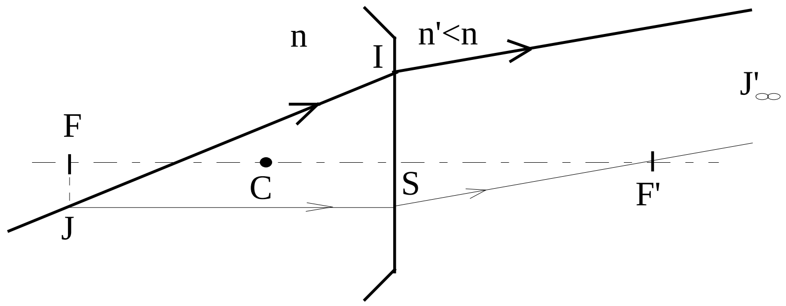


We can notice here that F and F' are symmetric with respect to the center of SC, which can be easily verified by a calculation. We can also see that F and F' are necessarily outside of SC. Finally we notice that for a converging surface, the center is in the medium with the larger index.

d) constructions

We will see a few examples of constructions; others will be done as exercises. The principle is always the same: we use specific rays, an incident ray parallel to the axis, an incident ray passing through F, an incident ray passing through C, an incident ray passing through S (be careful this last one is deflected by the spherical surface with an angular magnification n'/n).

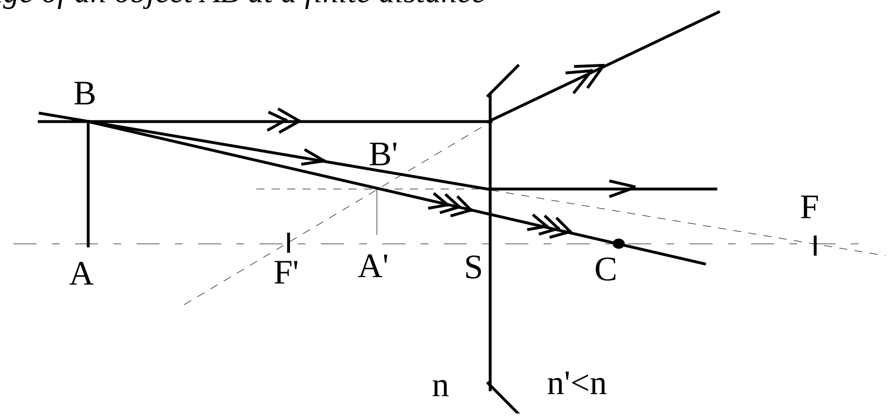
** emerging ray corresponding to an arbitrary incident ray*



converging concave
surface because n'<n

To construct the emerging ray, it is sufficient to construct the image of two specific points on the incident ray. Here we have chosen a point I on the surface, whose image I' is I itself, and a point J in the first focal plane whose image J' is at infinity in a direction obtained using the ray drawn in a thin line. The emerging ray passes through I'=I and J' at infinity (i.e. is parallel to the emerging ray in thin line).

**image of an object AB at a finite distance*



diverging convex
surface because n'<n

e) Newton's formulae

We can easily get those using the construction of the image of the object AB using rays passing through the focal points.

From the one arrow ray: $g_y = \dfrac{\overline{A'B'}}{\overline{AB}} = \dfrac{\overline{SF}}{\overline{AF}} = -\dfrac{f}{\overline{FA}}$

From the two arrows ray: $g_y = \dfrac{\overline{A'B'}}{\overline{AB}} = \dfrac{\overline{F'A'}}{\overline{F'S}} = -\dfrac{\overline{F'A'}}{f'}$

And finally: $\overline{FA}.\overline{F'A'} = f\,f'$

Review of the main results on refractive surfaces

* the plane refractive surface is stigmatic only for an object at infinity or on its surface;

* in the linear approximation, you must be able to recover rapidly all the properties of the plane refractive surface using constructions and the linear form of the refraction law. A figure allows you to recover that the transverse magnification is equal to 1 for all objects, the other magnifications can be found using Lagrange-Helmholtz's theorem and $g_y = g_x . g_\alpha$.

* a plane and parallel plate does not deflect rays, it induces a pure translation on a parallel beam.

* a plane and parallel plate brings the image closer by a quantity $e(n-1)/n$ from the position of the object, and this is independent on the distance between the object and the plate; all the magnifications are equal to 1.

* the spherical refractive surface can be converging or diverging depending on its curvature and on the ratio of the two indices; we can remember that it is converging if its center is in the medium with the higher index.

* the spherical refractive surface is stigmatic and aplanetic for its center of curvature, for its surface and for the Young-Weierstrass' points. For the latter, we can remember that their positions with respect to the center, for a ratio of indices n/n' and a radius of curvature $R = \overline{\mathrm{SC}}$, are given by *R n/n'* and *R n'/n*, and you can find which one is the object and which is the image with a drawing (using the fact that rays get closer to the normal if the index gets larger). You can also recover their positions rapidly knowing approximately where they are and writing that the optical path must be constant for the two vertices of the sphere (see proof in chapter IV). You should also be able to find their transverse magnification rapidly.

* for the spherical surface, as for all focal systems in the linear approximation, constructions must be perfectly mastered. You should always start by putting down the positions of the important points: S, C and the focal points. You must be able not only to construct the 4 specific rays that allow you to construct the image of an object at a finite distance, but also construct what happens when an object (resp. an image) is in the first (resp. second) focal plane, find the object knowing the position of the image, and construct the emerging ray knowing the incident ray and vice-versa.

* for the spherical surface, constructions allow you to recover all the formulae very rapidly knowing the approximate look of these formulae and using the following basic rules:

- Newton's formulae are found using rays passing through the focal points;
- among the gaussian formulae, it is best to remember only the one with the origin at the vertex S; it can be useful to remember it by heart, but to always check the signs on a drawing;
- the power, which is equal to n'/f' for any system in the linear approximation, is equal to (n'-n)/R for the spherical surface; it is always better to check the signs on a construction;
- the first and second focal points are symmetric with respect to the center of SC;
- when doing ray constructions, be careful about the ray passing through S, which is deflected according to the linearized law for refraction;
- the transverse magnification is easy to find with a construction, you get the others ones using the usual laws between magnifications.