CHAPTER VII

Study of simple elements: THE LENSES

I. Introduction. Description of a lens.

In general we will call lens any focal system consisting of two refractive surfaces (the systems with more surfaces are usually called objectives or eyepieces or another name referring to their function).

Almost all lenses that we will consider here are made of a homogeneous and isotropic medium limited by two surface, one of which at least is not plane. They will usually be used with the same index on both sides, air in most cases.

We will start this chapter with the study of thin lenses in the gaussian approximation. In practice, a thin lens is made of two spherical surfaces with their vertices very close to each other, and it is characterized by its focal length. In addition, we have seen, in the gaussian approximation, that any focal system could be equivalent to a thin lens, if you then translate the image space by the quantity HH' with respect to the object space. Mastering the gaussian approximation for thin lenses is thus particularly important.

We will then study the case of thick lenses. We will first study the conditions for stigmatism for these lenses. In the linear approximation, thick lenses are not simple elements anymore and we will study them using a general method that can be applied to any centered focal system, using its cardinal points (see chap. V). This will be an introduction to the study of more complex centered systems that we will see in the following chapters.

Finally we will end this chapter by reviewing other types of lenses meant for specific applications: Fresnel lenses, cylindrical lenses, gradient index lenses.

II. Thin lenses

1) Definition

A thin lens can be described from two points of view, but in any case you must be in the gaussian approximation.

A first description of a thin lens is a focal optical system with both principal planes on the lens itself, and with a well-known focal length f'. We will consider here lenses immersed in air on both sides. For the other situations, go back to the general properties of centered systems in the gaussian approximation.

It is also useful to connect the definition of a thin lens to its geometrical shape. A thin lens has a meaning only in the gaussian approximation, in which case the refractive surfaces that limit it are necessarily plane or spherical. We shall justify later that a lens can be considered thin if the thickness at its center (distance between the vertices of the two surfaces) is very small compared to the absolute values of the radii of curvature of the two surfaces, and also compared to the difference between those two radii (taken with their sign).

We will see now the link between those two points of view, and then we will use the definition in terms of focal length to study the formation of images by a thin lens.

2) Relation between geometrical shape and focal length for a thin lens

Let us consider the general case of a lens with index *n* immersed in air, consisting of two spherical surfaces with vertices S_1 and S_2 and centers C_1 and C_2. We will name $R_1 = \overline{S_1C_1}$ and $R_2 = \overline{S_2C_2}$ the algebraic radii of curvature of the spherical surfaces, measured positively in the direction of propagation of light. The thin lens approximation amounts to consider that S_1 and S_2 are superimposed and at the center O of the lens. The second focal length of the lens will then be $f' = \overline{OF'}$.

The second focal point F' of the lens will be the image of the second focal point F'_1 of the first refractive surface through the second surface, so if we write the gaussian formula for the second spherical surface:

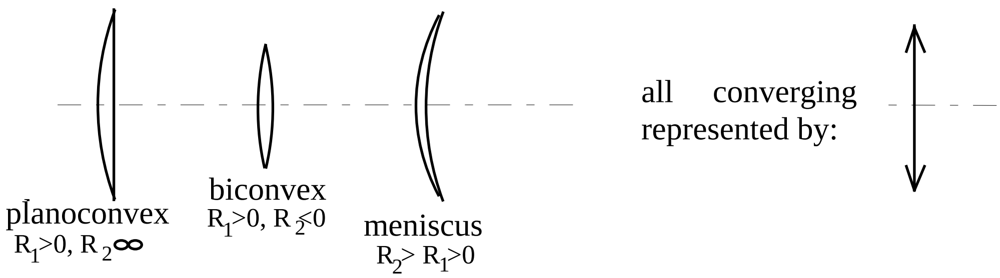
$$\frac{1}{\overline{S_2F'}} = \frac{n}{\overline{S_2F'_1}} + \frac{1-n}{R_2}$$

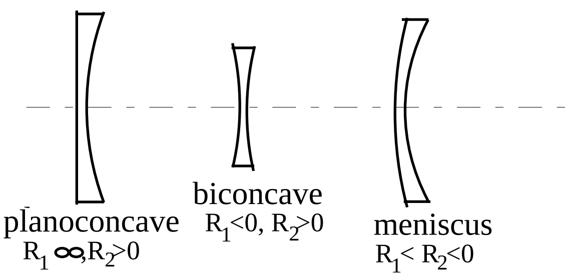
Using the fact that S_1 and S_2 are superimposed, we can see that the three terms of this equation represent the power of the lens, the power of the first surface and of the second surface. We can thus write this expression in the following way:

$$C = \frac{1}{f'} = \frac{n-1}{R_1} + \frac{1-n}{R_2} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

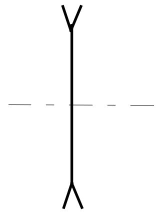
This important relationship for the second focal length of a thin lens in air shows that the focal length does not depend on the orientation of the lens (R_1 and R_2 are replaced by $R'_1 = -R_2$ and $R'_2 = -R_1$). In particular, a converging lens will remain converging for any orientation! This is in fact true of any optical system. However these properties can change completely if we change the indices of the incident and emerging media.

The following figure shows a few examples of lenses either converging of diverging, which can be considered as thin lenses:





All diverging represented by :

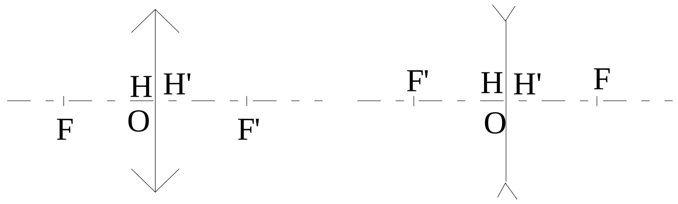


We can see on these drawings that for smaller radii of curvature of the refractive surfaces (smaller focal lengths), we must reduce the diameter of the lenses to remain in the thin lens approximation.

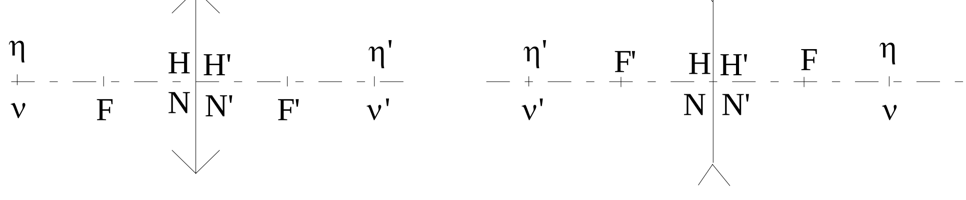
3) Cardinal points

We will consider here a lens immersed in air.

The position of all cardinal points can easily be found knowing the position of the center O of the lens and its second focal length (positive for a converging lens, negative for a diverging one). The principal points are both in O. The first and second focal points are symmetric with respect to O and at a distance *f* ' from O: they are both real for the converging lens and both virtual for the diverging lens. We thus have:



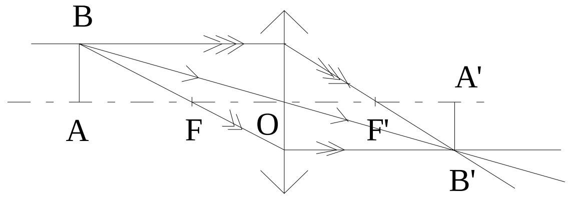
The other cardinal points can be deduced from these using the general properties of centered systems. Since the extreme media have the same index, principal points and nodal points are identical, so are antiprincipal and antinodal points (we can see it also using Lagrange-Helmholtz’s theorem which is written as $y\alpha = y'\alpha'$: if the transverse magnification is equal to 1, then the angular magnification is also 1, and the same thing is true for −1). In addition the antiprincipal points are symmetric from the principal points with respect to the corresponding focal points. We thus have:



4) Constructions

The specific rays which are useful for constructions are the incident ray parallel to the axis emerging through F', the incident ray passing through F emerging parallel to the axis, the ray passing through O which is not deflected (assuming still that the extreme media have the same index).

Here is an example of construction for a converging lens and a real object located to the left of F:



5) Position of the image. Magnifications.

a) gaussian formula (origin at the center O of the lens)

This is the general gaussian formula with the origin taken at the principal plane which is written here as:

$$\boxed{\frac{1}{x'}-\frac{1}{x}=\frac{1}{f'}} \quad \textit{with} \;\; x=\overline{\mathrm{OA}},\; x'=\overline{\mathrm{OA'}} \;\textit{and}\; P=1/f'$$

b) magnifications with the origin at O

The transverse magnification is easy to find using the ray passing through O:

$$\boxed{g_y=\frac{x'}{x}}$$

The other magnifications can be calculated using Lagrange-Helmholtz's theorem and the relationship $g_y = g_x . g_\alpha$:

$$g_\alpha=\frac{1}{g_y}=\frac{x}{x'} \qquad\qquad g_x=\frac{g_y}{g_\alpha}=(g_y)^2=\frac{x'^2}{x^2}$$

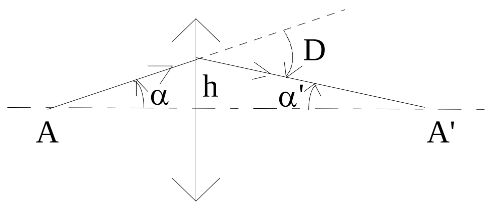
c) Newton's formulae

They can be found using the rays passing through the focal points:

$$\boxed{\begin{gathered} g_y=-\frac{f}{\overline{\mathrm{FA}}}=-\frac{\overline{\mathrm{F'A'}}}{f'} \\ \overline{\mathrm{FA}}.\overline{\mathrm{F'A'}}=ff'=-f'^2=-f^2 \end{gathered}}$$

6) Deflection induced by a lens

A ray making an angle α with the axis will be deflected by a lens, by an angle D:

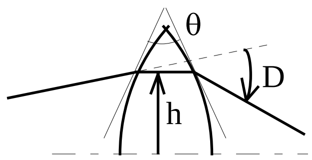


$$D=\alpha'-\alpha$$

We can write D as a function of the positions of points A and A' which are conjugate and of the height h of the ray on the lens:

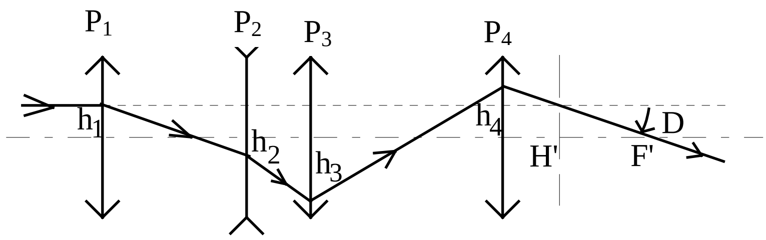
$$D=-\frac{h}{x'}+\frac{h}{x}=-\frac{h}{f'}=-hP$$

We easily find that this deflection depends only on the height of the impact point on the lens. We can interpret this in the following way: at height h, the thin lens acts as a prism with a small angle θ, inducing a deflection $D=(n-1)\theta$ which is independent on the angle of incidence:



We can use this property of lenses if we want to deflect a ray (or a thin parallel beam) by a small adjustable quantity: we use a long focal length lens which we translate perpendicular to the optical axis.

This property is also useful to figure out the focal length of a system consisting of several thin lenses. Let us consider the path of an incident ray parallel to the axis through those lenses:



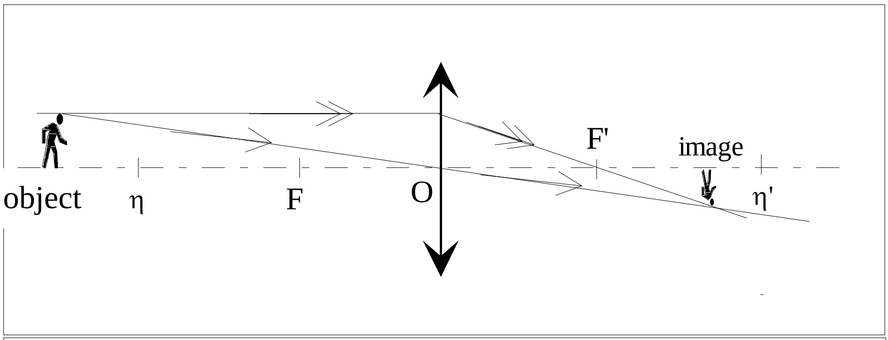
The total deflection D induced by all the lenses is the sum of the deflections of each lens:

$$D=-\sum_i h_i P_i$$

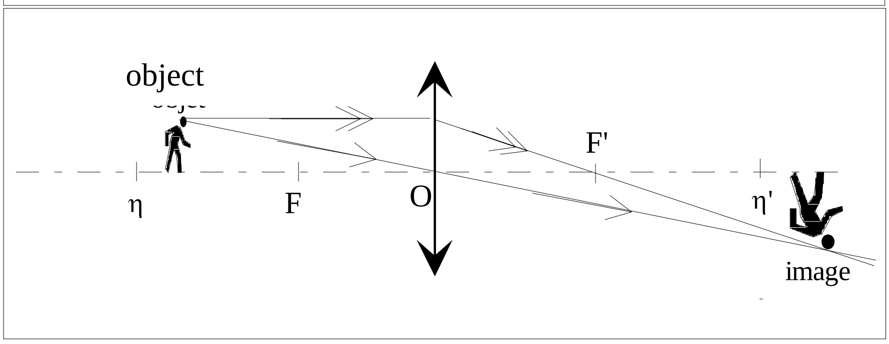
In addition we can see on the figure that $D=-h_1 P$, where P is the power of the whole system, which can thus be written as:

$$P=\frac{1}{h_1}\sum_i h_i P_i$$

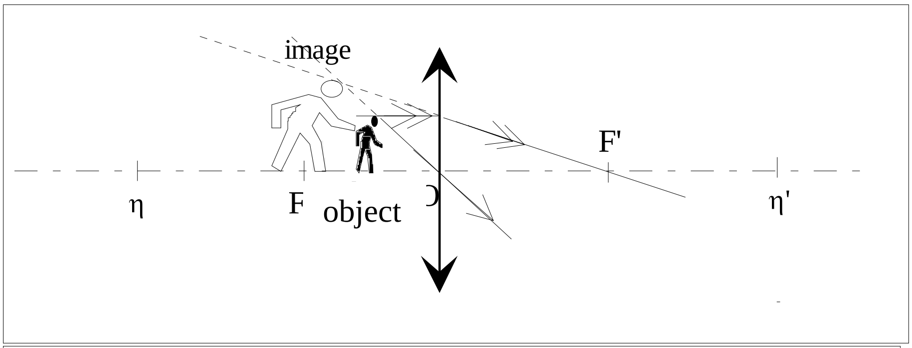
6) Variation of the position of the image when the object is moving along the axis



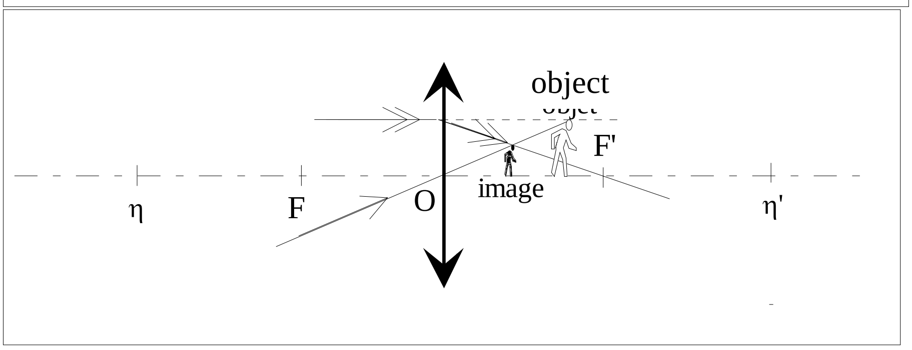
The object is real, located at more than twice the focal length from the center of the lens: its image is real, smaller and inverted.



The object is real, located between the first antiprincipal point and the first focal point: its image is real, larger and inverted.



The object is real, located between the first focal point and the lens: its image is virtual, larger and upright. (lens used as a magnifying glass)



The object is virtual: its image is real, smaller and upright.

The same type of constructions for a diverging lens (which should be done as an exercise) shows that the image is virtual at first, smaller and upright, for any real object; it becomes real, larger and upright for a virtual object located between the lens and the first focal point; then it becomes virtual and inverted when the virtual object is further downstream from the first focal point, larger than the object if it is located between the first focal and first antiprincipal points, smaller if the object moves further downstream from the first antiprincipal point.

Using such constructions, we can easily prove the following useful properties:

– to make a real image from a real object, you need a converging lens with a focal length smaller than a quarter of the distance d between the two points;

– if the above condition is fulfilled, there are two positions for the lens to conjugate the two planes : those are called the Bessel positions, symmetric with respect to the middle of the two conjugate planes ; in the case when d is exactly equal to 4 times the focal length, those two positions are identical.

– a converging lens always gives a real image of a virtual object; to get a virtual image for a virtual object, you need a diverging lens.

III. Thick lenses

We now want to study lenses in the general case when we cannot neglect anymore the distance between the two refractive surfaces. We should of course find that the results found for a thick lens should be the same of the ones for the thin lens if this thickness goes to zero. When the thickness is not negligible, it is not necessary to consider only the paraxial approximation, so we shall first study the stigmatic conditions for thick lenses.

1) Perfect stigmatism for lenses

a) Perfectly stigmatic single lenses

They are made of refractive surfaces used for specific points for which they are stigmatic. According to what we saw in chapter IV, we can use the following combinations:

– a plane surface if the object is at infinity (image at infinity also);

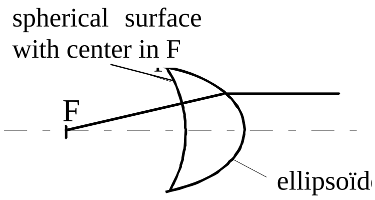
– a spherical surface for its center of curvature;

– a spherical surface for Young-Weierstrass' points;

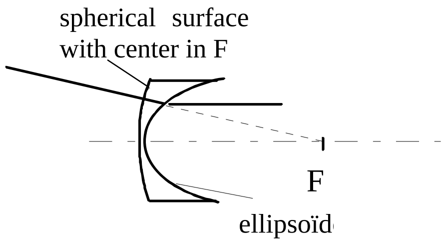
– Elliptical or hyperbolic surfaces for the couple infinity-focal point (and vice versa);

– the Descartes ovals for two points at a finite distance.

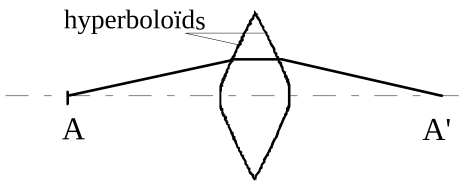
Combining all these surfaces, we can make the following stigmatic lenses:



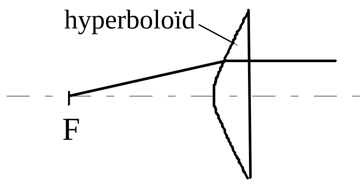
Sphero-elliptic lens
converging



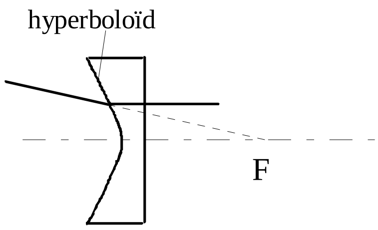
Sphero-elliptic lens
diverging



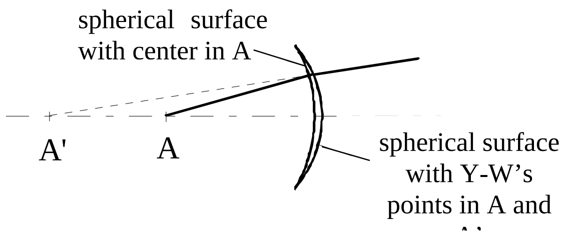
double-hyperbolic lens
(converging)



plano-convex hyperbolic lens
(converging)



plano-concave hyperbolic lens
(diverging)



Converging meniscus

Note that there are other possibilities, using the Descartes' ovals. Aspheric surfaces in general are much more difficult to make than plane or spherical surfaces. In situations when stigmatism is important, it is often preferred to make systems with several lenses with spherical surfaces, with aberrations that compensate each other. However, due to improvement in the fabrication methods, aspheric lenses are more widely used, either made of glass or of molded plastic, for some applications when the number of surfaces muse be limited to reduce the intensity losses (condensers for lighting systems or projection systems) or to reduce the weight (diode laser collimation in optical reading systems, disposable cameras).

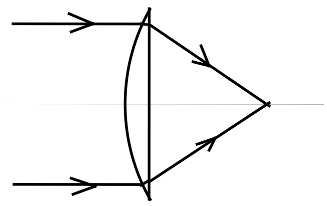
b) best orientation of single lenses

Even though there are not perfectly stigmatic, lenses with spherical surfaces induce more or less aberrations (in particular spherical aberration) depending on their shape and the conjugate points there are used for. We will not study here the theory of aberrations (this will be done in the optical design course); we will only mention a few simple rules about the best way to use single lenses to minimize aberrations.

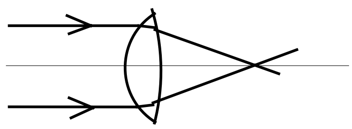
One simple idea underlying these rules is that in order to minimize aberrations you should reduce the angles of incidence of the surfaces. In the case when the incidence angle varies in opposite direction for the two surfaces, the best situation is when the two angles are equal: in other words, if we consider the prism that is locally equivalent to the lens, we try to use it at the deviation minimum.

What is useful to remember at this point is:

→ for an object at infinity, the best form spherical lens is a biconvex lens with $R_2 = -6R_1 < 0$; a single plano convex lens is almost as good, if you use it with its plane surface towards the focal point:



Best orientation
of a plano convex lens

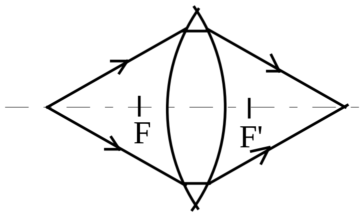


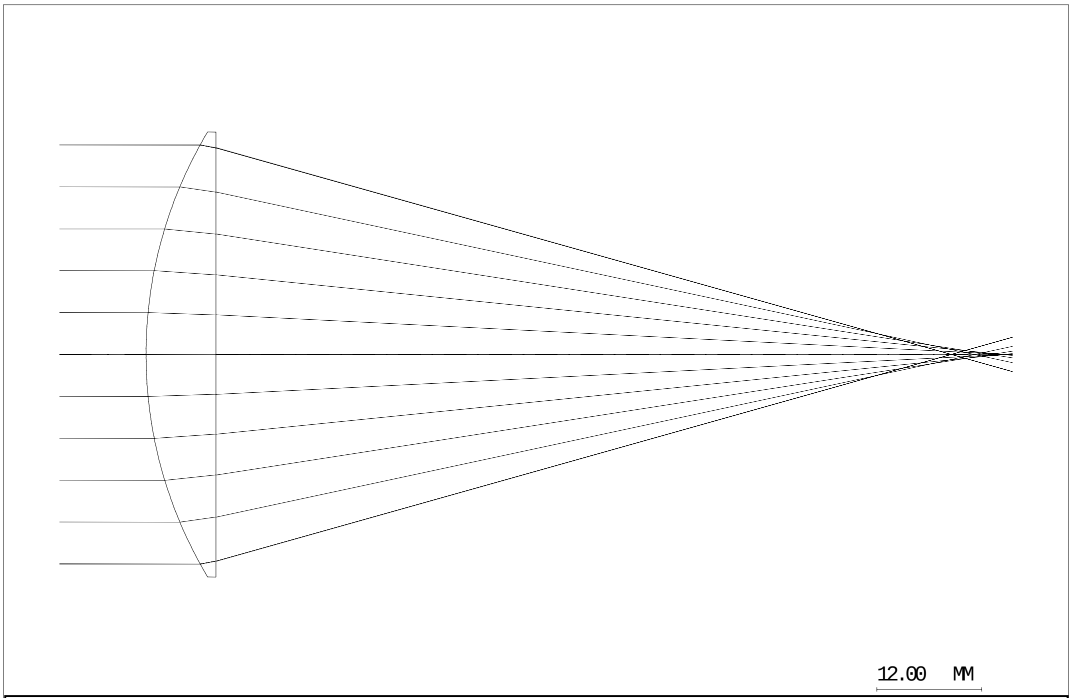
best form lens

The two ray tracings shown on the next page illustrate the best orientation of a plano convex lens for these conjugate points.

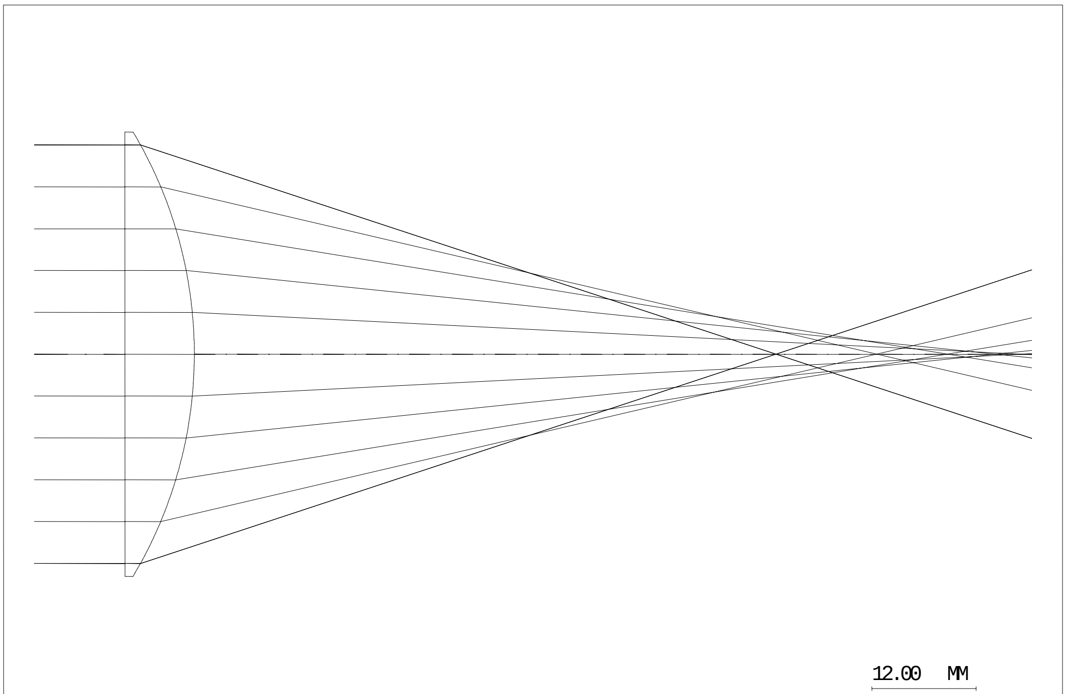
More generally, for any lens shape, the best orientation is obtained for the most curved surface towards the object at infinity. Of course the same type of arguments can be applied symmetrically to the case of an image at infinity.

→ for symmetric conjugate points with distances 2f/2f, the best lenses is a symmetric biconvex lens with $R_2 = -R_1$:





Plano convex lens used with the best orientation

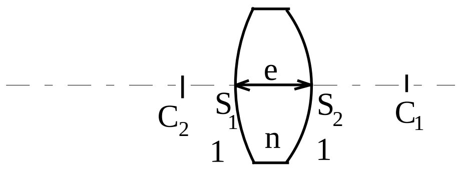


Plano convex lens used in the wrong orientation

2) Thick lenses in the linear approximation

a) characteristics

Since we only consider here the linear approximation, the surfaces limiting the lens will necessarily be spherical or plane. The optical axis of the lens is defined by the line joining the centers of the two spheres. The lens will in general be immersed in air. We will write R_1 and R_2 the algebraic radii of curvature of each surface; e the thickness on axis of the lens and n the index of the material the lens is made of.



We will often characterize a lens by its power P in diopters (m^{-1}), which for a lens immersed in air is the inverse of its image focal length. The power of a lens (as for any centered system) is independent on the orientation (i.e. on the direction of propagation of light). We shall see next how this power relates to the geometrical properties of the lens. Note that it is not sufficient to define completely lens, even in the linear approximation.

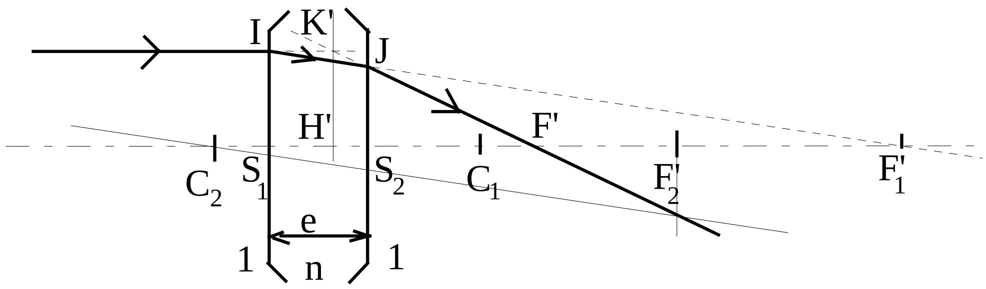
Another useful parameter is the following:

$$\gamma = \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$

As opposed to the power this parameter changes sign when we change the direction of propagation of light through the lens. For example a plano convex lens has a positive γ when it is used in the best orientation for an object at infinity (curved surface towards the object) and negative for the wrong orientation.

b) determination of the focal length

We shall calculate it using a construction using an incident ray parallel to the optical axis.



F’ can be found knowing that it is the conjugate of F'$_1$ through the second surface:

$$\frac{1}{\overline{S_2F'}}=\frac{n}{\overline{S_2F'_1}}+P_2$$

where P_2 is the power of the second surface.

Using the triangles F'$_1$S$_1$I and F'H'K', we can write:

$$\frac{\overline{S_1I}}{\overline{S_1F'_1}}=\frac{\overline{S_2J}}{\overline{S_2F'_1}} \qquad \frac{\overline{S_1I}}{\overline{H'F'}}=\frac{\overline{S_2J}}{\overline{S_2F'}}$$

For this we can get:

$$\frac{1}{\overline{H'F'}}=\frac{\overline{S_2F'_1}}{\overline{S_1F'_1}}\cdot\frac{1}{\overline{S_2F'}}$$

Using the relation giving the position of F', we obtain:

$$\frac{1}{\overline{H'F'}}=\frac{n}{\overline{S_1F'_1}}+\frac{\overline{S_2F'_1}}{\overline{S_1F'_1}}\cdot P_2=P_1+P_2-\frac{e}{n}P_1P_2$$

where P_1 and P_2 are the powers of each surface. Note that the above relation gives the power of the whole lens as a function of the powers of each of its elements: it is called the Gullstrand’s formula, and we will see it again later in a more general case in the following chapter relative to the combination of centered optical systems.

If we write this relation as a function of the radii of curvature of the two surfaces of the lens, we get:

$$\frac{1}{f'}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)+\frac{e(n-1)^2}{nR_1R_2}$$

c) thin lens approximation

We see in the above expression of the focal length of the lens that if the thickness e goes to zero, we find the expression of the focal length of a thin lens. More precisely, it requires that the additional term $e(n-1)^2/nR_1R_2$ be negligible compared to the other two, which leads to the following condition:

$$e << |R_1 - R_2|$$

This condition implies that the focal length of the lens is the same as that of a thin lens with the same radii of curvature. However this condition is not sufficient to imply that the two principal planes of the lens are superimposed: it is not a condition sufficient for the lens to be equivalent to a thin lens.

Note in particular the case when one of the surfaces is plane (R_1 or R_2 infinite): the focal length of the lens is then independent of the thickness *e* at its center. The focal length of such a lens is thus the same as that of a thin lens, but the principal planes are separated by a distance *e*' which increases with *e*.

d) relation between the thickness on the edge of the lens and its power

Let us calculate the thickness at a distance *h* from axis, *e*(*h*), of a lens with a known thickness at the center *e*=*e*(0) and with radii of curvature R_1 and R_2 (with a sign depending on the concavity of each surface). Keeping only the second order terms in *h*, we get:

$$e(h)=e-\frac{h^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

For a thin lens, the variation in thickness from center to edge is proportional to the power of the lens:

$$e(h)-e(0)=-\frac{h^2}{2(n-1)}P$$

For a converging thin lens (*P*>0), the thickness is thus smaller on the edge than at its center while it is the opposite for a thin diverging lens: we get the same converging properties of the different types of lenses that we saw in the previous paragraph.

For thicker lenses, the relationship between variation of thickness and convergence is not so straightforward. However, this property remains valid as long as the thickness *e* is not too large (as an exercise try to find a counter example!).

e) determination of cardinal points

Several methods allow us to find the focal points and principal points of a thick lens, for which we can determine all the other cardinal points. In fact, it is sufficient to determine only three of these points, for example F, F' and H', since we have the following relationship:

$$\frac{\overline{\mathrm{H'F'}}}{\overline{\mathrm{HF}}}=\frac{f'}{f}=-\frac{n'}{n}$$

* *graphical method*

We place on a drawing, preferable to scale, the vertex, center and focal points of each surface. We construct the path of an incident ray parallel to the optical axis, for which we get the second focal point of the system and its second principal plane. Then we construct the inverse path of a ray that emerges from the system parallel to the axis, from which we get F and H. We can then check that the relationship between $\overline{\mathrm{HF}}$ and $\overline{\mathrm{H'F'}}$ is fulfilled, which gives a confirmation of the accuracy of the constructions.

The imprecisions of the construction are such that it is best not to measure directly on it the positions of the cardinal points. The graphical method is rather an indispensable companion to the calculation of these positions.

** use of the conjugation formulas*

This is the way we used earlier to calculate the value of the focal length. We calculate the position of the image of an object at infinity on axis through the two surfaces, which amounts to calculate the image of F'_1 through the second surface, and we get the position of F' with respect to the cardinal points of the surfaces.

To calculate the position of H', we can proceed in two ways. We have seen a method using the properties of the graphical construction. We can also determine the power of the lens as a function of the powers of the two surfaces (Gullstrand's formula $P=P_1+P_2-(e/n)P_1P_2$), from which we get $\overline{H'F'}=n'/C$.

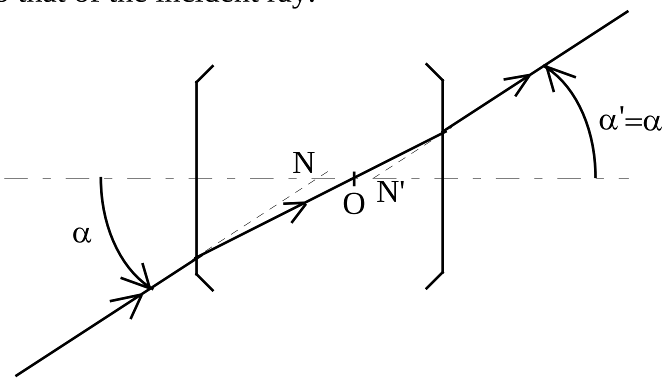
To calculate the position of F, we can proceed in the same way as for F': we calculate the position of the object that gives an image at infinity through the lens, which amounts to determining the object that gives an image through the first surface in F_2, first focal point of the second surface.

** matrix method*

This method is only appropriate for complex systems made of many surfaces. It can be done as an exercise and the general principles of this method will be described in annex of the chapter 8 relative to the combination of centered systems.

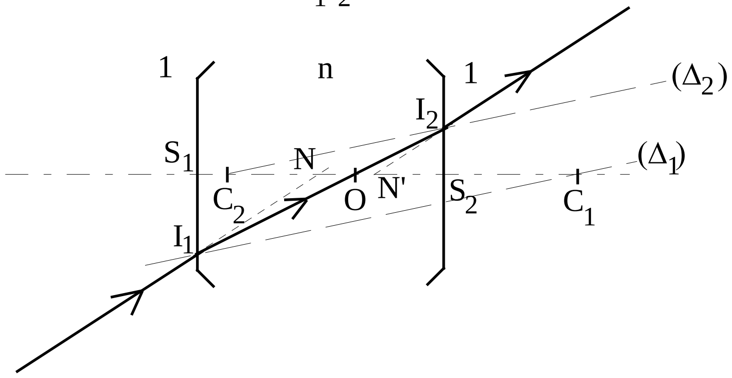
f) optical center of a thick lens

It can be interesting to determine the cardinal points of a lens from the nodal points rather than the principal points. In fact, when the extreme media have the same index, nodal points and principal points are identical. Let us consider an incident ray passing through the first nodal point N. It will emerge from the lens passing through the second nodal point N', in a direction parallel to that of the incident ray.



The **optical center O of the lens** is the point when the intermediate ray crosses the optical axis. It is thus the image of N through the first surface, and its image through the second surface is N'.

The interest of this optical center lies in the fact that it is easy to determine graphically. Let us draw two parallel lines, one Δ_1 passing through the center of the first surface C_1, the other one Δ_2 passing through C_2. Note I_1 the intersection of Δ_1 with the first surface and I_2 the intersection of Δ_2 with the second surface. C_1I_1 represents the normal to the first surface at I_1 and C_2I_2 the normal to the second surface at I_2. The ray I_1I_2 has thus the same angle of incidence on each surface, and the two corresponding incident and emerging rays are thus parallel. The intersection of I_1I_2 with the axis is in O.



This graphical determination of point O allows to calculate its position, using the similar triangles $C_1S_1I_1$ and $C_2S_2I_2$ on one hand, and OS_1I_1 and OS_2I_2 on the other hand:

$$\frac{\overline{I_1S_1}}{\overline{I_2S_2}} = \frac{\overline{C_1S_1}}{\overline{C_2S_2}} = \frac{\overline{OS_1}}{\overline{OS_2}}$$

We see that point O divides the segment S_1S_2 in two parts with lengths in the ratio R_1/R_2. We can write the position of O as a function of the thickness at the center of the lens $e = \overline{S_1S_2}$ and of the radii of curvature in the following way:

$$\overline{S_1O} = \frac{e}{1 - R_2/R_1}$$

We can deduce from it the positions of N and N' using the conjugation formulas for each surface. We find of course that in the case of a symmetric lens ($R_1 = -R_2$), the optical center is in the middle of S_1S_2.

g) constructions and conjugation formulas

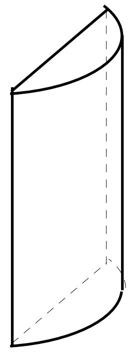
Once we have determined the positions of the focal points and the principal points using one of the above methods, we can determine all the other cardinal points then make any construction as outlined in the general methods for centered systems in the linear approximation (see chapter 5). We can also use the conjugation formulas relative to the principal planes or to the focal points, as well as the expressions of the magnifications.

IV. Other types of lenses

We will only mention here the existence of other types of lenses, without studying them in detail.

1) Cylindrical lenses

They are made of two surfaces that consist of portions of cylinders instead of spheres. For example, a cylindrical plano-convex lens will have the following shape:



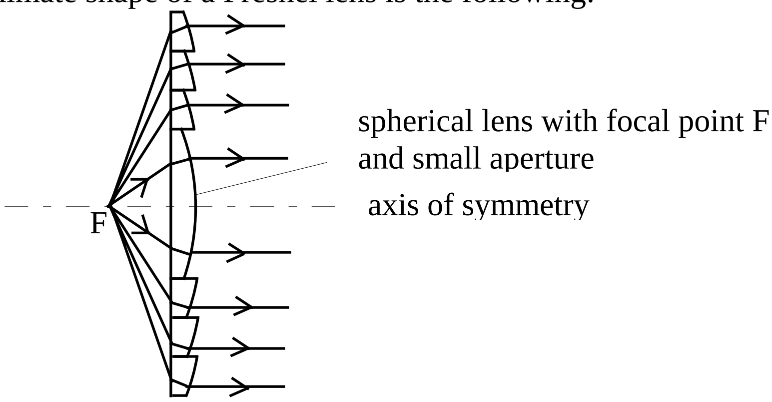
These lenses are no longer symmetric around one axis but have instead a plane of symmetry. For rays propagating in that plane of symmetry, the lens is equivalent to a parallel plate. For rays propagating in a plane orthogonal to that plane of symmetry, the lens will be equivalent to a spherical lens. The image of a point at infinity through such a lens will thus be a line, parallel to the generatrix of the cylinder.

This kind of lens is used when you want to transform a beam with a circular section into a beam with an elliptical section, or vice-versa. For example it can be used to circularize the output beam of a laser diode, which is in general asymmetrical.

2) Fresnel lenses

This kind of lens is used when a very short focal length is required with a very wide field of view (lenses placed on the back windshield of a bus, or at the counter of supermarkets) or to collect efficiently the light from a source (collimating lens of lighthouses). In these conditions, a traditional lens would present a lot of aberrations because the angles of incidence are very large.

The approximate shape of a Fresnel lens is the following:

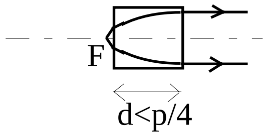


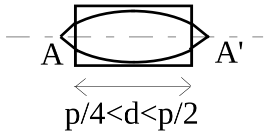
The principle consists in cutting the lens in annular sectors, with a prismatic shape suited to the deviation that we want to produce at the height *h* of the rays.

3) Gradient index lenses

We can understand the principle of these lenses by considering them as sections of optical fibers (with a larger diameter than actual fibers). They come as cylinders with plane extremities, perpendicular to their axis, with an index that decreases from the center to the edges. If the index variation is well chosen, the rays will propagate along a sinusoidal curve inside the gradient index material.

Through an appropriate choice of the length of the lens with respect to the period p of the sinusoid, we can produce lenses that will perform the conjugation between infinity and a focal point, or between two points at a finite distance. Here are two examples:





This kind of lens gives access to a short focal length in a simple way, with a small lens and without too much aberration. They are often used to collimate diode lasers.