Parametric amplification

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Introduction

Let us consider 3 waves at ω_1, ω_2, ω_3

propagating in the same direction

$$\mathcal{E}_j(z,t) = \hat{\mathrm{e}}_j A_j(z,\omega_j) e^{i(k_j z - \omega_j t)} + C.C \qquad \text{with } j = 1,2,3$$

$$\omega_1 + \omega_2 = \omega_3$$

Assumption: isotropic medium

Parametric approximation

$\mathcal{E}_1(0) \neq 0,$

$A_3(z, \omega_3) = A_3 = cte$

Coupled wave equations

$$\frac{dA_1}{dz} = \frac{i\omega_1}{n_1 c}\chi_{eff}^{(2)} A_3 A_2{}^* e^{i\Delta k z} \quad (1)$$

$$\frac{dA_2}{dz} = \frac{i\omega_2}{n_2 c}\chi_{eff}^{(2)} A_3 A_1{}^* e^{i\Delta k z} \quad (2)$$

with $\Delta k = k_3 - k_1 - k_2$

$$\chi_{eff}^{(2)} = \hat{\mathbf{e}}_1 \underline{\underline{\underline{\chi}}}^{(2)}(\omega_3, -\omega_2)\hat{\mathbf{e}}_3\hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_2 \underline{\underline{\underline{\chi}}}^{(2)}(\omega_3, -\omega_1)\hat{\mathbf{e}}_3\hat{\mathbf{e}}_1$$

Coupled wave equations

$$\begin{cases} \dfrac{dA_1}{dz} = \alpha_1 A_2^* e^{i\Delta kz} \quad \text{where } \alpha_1 = \dfrac{i\omega_1}{n_1 c}\chi_{eff}^{(2)} A_3 \\ \dfrac{dA_2}{dz} = \alpha_2 A_1^* e^{i\Delta kz} \quad \text{where } \alpha_2 = \dfrac{i\omega_2}{n_2 c}\chi_{eff}^{(2)} A_3 \end{cases}$$

Coupled wave equations

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$$\frac{d^2A_1}{dz^2} = \alpha_1 \left(\frac{dA_2^*}{dz} + i\Delta k A_2^* \right) e^{i\Delta kz}$$

Uncoupling

$$\frac{d^2A_1}{dz^2} = \alpha_1 \left(\frac{dA_2^*}{dz} + i\Delta k A_2^* \right) e^{i\Delta kz}$$
$$= \alpha_1 \left[\alpha_2^* A_1 e^{-i\Delta kz} + \frac{i\Delta k}{\alpha_1} \left(\frac{dA_1}{dz} e^{-i\Delta kz} \right) \right] e^{i\Delta kz}$$

$$\frac{d^2A_1}{dz^2} = +i\Delta k \frac{dA_1}{dz} + \alpha_1 \alpha_2^* A_1$$

Characteristic equation

$$R^2 - i\Delta k R - \alpha_1 \alpha_2^* = 0$$

$$\Delta = -\Delta k^2 + 4\alpha_1\alpha_2^* = -\Delta k^2 + 4\gamma_0^2$$

where

$$\gamma_0^2 = \frac{\omega_1 \omega_2}{n_1 n_2} \frac{\left|\chi_{eff}^{(2)}\right|^2 |A_3|^2}{c^2}$$

Threshold

$$\Delta = -\Delta k^2 + 4\left(\frac{2\pi\left|\chi_{eff}^{(2)}\right|\,|A_3|}{\sqrt{n_1\lambda_1 n_2\lambda_2}}\right)^2 > 0$$

$\Delta = 4\gamma^2$

$$R_1 = i\frac{\Delta k}{2} + \gamma$$

$$R_2 = i\frac{\Delta k}{2} - \gamma$$

Solutions

$$A_1(z) = e^{+i\frac{\Delta k}{2}z}\left(Ae^{+\gamma z} + Be^{-\gamma z}\right)$$

$$\frac{dA_1}{dz} = \left[+i\frac{\Delta k}{2}\left(Ae^{+\gamma z} + Be^{-\gamma z}\right) + \gamma Ae^{\gamma z} - \gamma Be^{-\gamma z}\right] e^{+i\frac{\Delta k}{2}z}$$

$$A_2^*(z) = \frac{1}{\alpha_1}\left[+i\frac{\Delta k}{2}\left(Ae^{+\gamma z} + Be^{-\gamma z}\right) + \gamma Ae^{\gamma z} - \gamma Be^{-\gamma z}\right] e^{-i\frac{\Delta k}{2}z}$$

Boundary conditions

$$\begin{cases} A_1(0) = A + B \\ A_2^*(0) = \frac{1}{\alpha_1}\left[(+i\frac{\Delta k}{2} + \gamma)A - (-i\frac{\Delta k}{2} + \gamma)B\right] \end{cases}$$

$$\begin{cases} A = \frac{1}{2\gamma}\left[\alpha_1 A_2^*(0) + \left(-i\frac{\Delta k}{2} + \gamma\right) A_1(0)\right] \\ B = \frac{1}{2\gamma}\left[\left(i\frac{\Delta k}{2} + \gamma\right) A_1(0) - \alpha_1 A_2^*(0)\right] \end{cases}$$

Solutions

$$\boxed{A_1(z) = \left[A_1(0)\left(ch\gamma z - i\frac{\Delta k}{2\gamma}sh\gamma z\right) + \frac{\alpha_1}{\gamma}A_2^*(0)sh\gamma z\right]e^{+i\frac{\Delta k}{2}z}}$$

Solutions

$$A_2(z) = \frac{1}{\alpha_1^*}\left[\left(i\frac{\Delta k}{2}+\gamma\right)A^* e^{\gamma z} + \left(i\frac{\Delta k}{2}-\gamma\right)B^* e^{-\gamma z}\right] e^{-i\frac{\Delta k}{2}z}$$

$$= \frac{1}{\alpha_1^*}e^{-i\frac{\Delta k}{2}z}\left\{\left(i\frac{\Delta k}{2}+\gamma\right)\frac{1}{2\gamma}\left[\alpha_1^* A_2(0) + \left(-i\frac{\Delta k}{2}+\gamma\right)A_1^*(0)\right]e^{\gamma z}\right.$$

$$\left. + \left(i\frac{\Delta k}{2}-\gamma\right)\frac{1}{2\gamma}\left[\left(i\frac{\Delta k}{2}+\gamma\right)A_1^*(0) - \alpha_1^* A_2(0)\right]e^{-\gamma z}\right\}$$

$$A_2(z) = \left[i\frac{\Delta k}{2\gamma}A_2(0)sh\gamma z + A_2(0)ch\gamma z + \frac{4\gamma^2+\Delta k^2}{4\gamma\alpha_1 *}\right]e^{-i\frac{\Delta k}{2}z}$$

Solutions

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$$A_2(z) = \left[A_2(0)\left(ch\gamma z - i\frac{\Delta k}{2\gamma}sh\gamma z\right) + \frac{\alpha_2}{\gamma}A_1^*(0)sh\gamma z\right]e^{+i\frac{\Delta k}{2}z}$$

Pump threshold

$$I_3 = 2n_3\epsilon_0 c\left|A_3\right|^2$$

$$4\gamma_0^2 > \Delta k^2$$

$$4\frac{\omega_1\omega_2}{n_1 n_2}\frac{\left|\chi_{eff}^{(2)}\right|^2\left|A_3\right|^2}{c^2} > \Delta k^2$$

$$2n_3\epsilon_0 c\left|A_3\right|^2 > 2n_3\epsilon_0 c\frac{n_1 n_2 c^2}{4\omega_1\omega_2\left|\chi_{eff}^{(2)}\right|^2}\Delta k^2$$

$$\boxed{I_3 > \frac{\Delta k^2}{\left|\chi_{eff}^{(2)}\right|^2}\frac{n_1 n_2 n_3\epsilon_0 c^3}{2\omega_1\omega_2}}$$

Solutions : if $A_2(0) = 0$

The signal is amplified and the idler frequency is generated in the medium.

$$A_1(z) = e^{+i\frac{\Delta k}{2}z}\left[A_1(0)\left(ch\gamma z - i\frac{\Delta k}{2\gamma}sh\gamma z\right)\right]$$

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Noiseless amplification ?