Introduction to quantum optics

Fields quantization

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Introduction: Maxwell's equations

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

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Vector potential, in Coulomb Gauge

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$$\nabla \cdot \mathbf{A} = 0 \qquad \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

Transverse plane waves basis : mode concept

In the quantization volume:

$$\mathbf{A}^{(+)}(\mathbf{r},t) = \sum_{\ell} \mathcal{A}_\ell(t)\vec{\varepsilon}_\ell e^{\imath \mathbf{k}_\ell\cdot\mathbf{r}}$$

ℓ mode corresponds to a wave vector $\mathbf{k} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}\frac{2\pi}{L}$ and one of the two orthogonal transverse polarisations $\vec{\varepsilon}_m$.

$$\mathcal{A}_\ell(t) = \frac{1}{L^3}\int_{L^3} d^3\mathbf{r}\,\mathbf{A}^{(+)}(\mathbf{r},t)\cdot\vec{\varepsilon}_\ell e^{-\imath \mathbf{k}_\ell\cdot\mathbf{r}}$$

$$\begin{aligned}\frac{d^2\mathcal{A}_\ell(t)}{dt^2} &= -c^2\left|\mathbf{k}_\ell\right|^2\mathcal{A}_\ell(t)\\ &= -\omega_\ell^2\mathcal{A}_\ell(t) \qquad \text{where } \omega_\ell = c\left|\mathbf{k}_\ell\right|\end{aligned}$$

Solutions : harmonic waves

$$\iota\hbar \frac{d\mathcal{A}_\ell(t)}{dt} = \hbar\omega_\ell \mathcal{A}_\ell(t)$$

$\mathcal{A}_\ell(t) = \mathcal{A}_\ell(0)e^{-\iota\omega_\ell t}$

Complex fields

$$\mathbf{E}^{(+)} = \sum_\ell \mathbf{E}_\ell^{(+)} = \iota \sum_\ell \omega_\ell \vec{\varepsilon}_\ell \mathcal{A}_\ell(t) e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}}$$

$$\mathbf{B}^{(+)} = \sum_\ell \mathbf{B}_\ell^{(+)} = \iota \sum_\ell (\mathbf{k}_\ell \times \vec{\varepsilon}_\ell) \mathcal{A}_\ell(t) e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}}$$

Energy of the electromagnetic field

$$u_\ell = \frac{1}{2}\left(\epsilon_0 \mathbf{E}_\ell^2 + \frac{1}{\mu_0}\mathbf{B}_\ell^2\right)$$

$$\begin{aligned}
\mathcal{H} &= \int_{L^3} d^3\mathbf{r} \sum_\ell \langle u_\ell \rangle \\
&= \int_{L^3} d^3\mathbf{r} \sum_\ell \langle \epsilon_0 \mathbf{E}^2 \rangle \\
&= \sum_\ell 2\epsilon_0 L^3 \omega_\ell^2 \left|\mathcal{A}_\ell\right|^2 \\
&= \sum_\ell \mathcal{H}_\ell
\end{aligned}$$

Conjugate canonical variables : definition

Hamilton-Jacobi equations

$$\frac{\partial \mathcal{H}}{\partial p_\ell} = \frac{dq_\ell}{dt} \qquad (1)$$

$$\frac{\partial \mathcal{H}}{\partial q_\ell} = -\frac{dp_\ell}{dt} \qquad (2)$$

The $(\hat{p}_\ell, \hat{q}_\ell)$ operators are then associated to the variables (p_ℓ, q_ℓ) with the commutation relation $[\hat{p}_\ell, \hat{q}_{\ell'}] = \hat{p}_\ell \hat{q}_{\ell'} - \hat{q}_{\ell'} \hat{p}_\ell = \imath\hbar\delta_{\ell\ell'}$

Conjugate canonical variables of the electromagnetic field

Let's have :

$$\mathcal{A}_{q\ell}(t) = \Re\{\mathcal{A}_\ell(t)\} \qquad \mathcal{A}_{p\ell}(t) = \Im\{\mathcal{A}_\ell(t)\}$$

so that $\mathcal{A}_\ell(t) = \mathcal{A}_{q\ell}(t) + \imath\mathcal{A}_{p\ell}(t)$

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$$\mathcal{H} = \sum_\ell 2\epsilon_0 L^3 \omega_\ell^2 \left(\mathcal{A}_{q\ell}^2 + \mathcal{A}_{p\ell}^2\right)$$

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$\iota\hbar\frac{d\mathcal{A}_\ell(t)}{dt} = \hbar\omega_\ell\mathcal{A}_\ell(t) \Rightarrow$

$$\frac{d}{dt}\mathcal{A}_{q\ell} = \omega_\ell\mathcal{A}_{p\ell}$$
$$\frac{d}{dt}\mathcal{A}_{p\ell} = -\omega_\ell\mathcal{A}_{q\ell}$$

Conjugate canonical variables of the electromagnetic field

$$\frac{d\mathcal{H}}{d\mathcal{A}_{q\ell}} = -4\epsilon_0 L^3 \omega_\ell \frac{d\mathcal{A}_{p\ell}}{dt}$$
$$\frac{d\mathcal{H}}{d\mathcal{A}_{p\ell}} = 4\epsilon_0 L^3 \omega_\ell \frac{d\mathcal{A}_{q\ell}}{dt}$$

$$\sqrt{4\epsilon_0 L^3 \omega_\ell}\mathcal{A}_{q\ell}(t) \qquad \text{et} \qquad \sqrt{4\epsilon_0 L^3 \omega_\ell}\mathcal{A}_{p\ell}(t)$$

The associated operators $\hat{\mathcal{A}}_{q\ell}(t)$ and $\hat{\mathcal{A}}_{p\ell}(t)$ must satisfy

$$\left[\sqrt{4\epsilon_0 L^3 \omega_\ell}\hat{\mathcal{A}}_{q\ell}, \sqrt{4\epsilon_0 L^3 \omega_{\ell'}}\hat{\mathcal{A}}_{p\ell'}\right] = \imath\hbar\delta_{\ell\ell'}$$

$$\hat{\mathcal{A}}_\ell = \hat{\mathcal{A}}_{q\ell} + \imath\hat{\mathcal{A}}_{p\ell}$$

Dimensionless operators

$$\hat{a}_\ell = \sqrt{\frac{2\epsilon_0 L^3 \omega_\ell}{\hbar}}\hat{\mathcal{A}}_\ell$$

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Quantized fields

$$\hat{\mathbf{E}}^{(+)} = \iota \sum_\ell \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 L^3}} \vec{\varepsilon}_\ell \, \hat{a}_\ell \, e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}} \qquad \hat{\mathbf{B}}^{(+)} = \iota \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 L^3 \omega_\ell}} (\mathbf{k}_\ell \times \vec{\varepsilon}_\ell) \, \hat{a}_\ell \, e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}}$$

Hamiltonian

$$
\begin{aligned}
\hat{\mathcal{H}} &= \sum_{\ell} 2\epsilon_0 L^3 \omega_\ell^2 \left(\hat{\mathcal{A}}_{p\ell}^2 + \hat{\mathcal{A}}_{q\ell}^2\right) \\
&= \sum_{\ell} \epsilon_0 L^3 \omega_\ell^2 \left(\hat{\mathcal{A}}_\ell \hat{\mathcal{A}}_\ell^\dagger + \hat{\mathcal{A}}_\ell^\dagger \hat{\mathcal{A}}_\ell\right) \\
&= \sum_{\ell} \frac{\hbar\omega_\ell}{2} \left(\hat{a}_\ell \hat{a}_\ell^\dagger + \hat{a}_\ell^\dagger \hat{a}_\ell\right) \\
&= \sum_{\ell} \hbar\omega_\ell \left(\hat{a}_\ell^\dagger \hat{a}_\ell + \frac{1}{2}\right) \\
&= \sum_{\ell} \hat{\mathcal{H}}_\ell
\end{aligned}
$$

N Operator

$\hat{N}_\ell = \hat{a}_\ell^\dagger \hat{a}_\ell$

$$\left[\hat{a}_\ell, \hat{a}_{\ell'}^\dagger\right] = \delta_{\ell\ell'}$$

$\hat{\mathcal{H}} = \sum_\ell \hbar\omega_\ell \left(\hat{N}_\ell + \frac{1}{2}\right)$

$$\hat{N}_\ell|n_\ell\rangle = n_\ell|n_\ell\rangle$$

The eigenvalues of $\hat{N}_\ell$ are quantized

$$\hat{\mathcal{H}}|n_\ell\rangle = \hbar\omega_\ell \left(n_\ell + \frac{1}{2}\right)|n_\ell\rangle$$

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Quantized energy

Photon= energy quantum

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Quantized energy

Photon= energy quantum

$$\hat{N}_\ell \Rightarrow \text{Photon number}$$

Photon creation and annihilation

$$\hat{a}_\ell |n_\ell\rangle = \sqrt{n_\ell}\,|n_\ell - 1\rangle$$
$$\hat{a}_\ell^\dagger |n_\ell\rangle = \sqrt{n_\ell + 1}\,|n_\ell + 1\rangle$$

The number states form a basis of the Fock space

$$\langle n_\ell | n_{\ell'}\rangle = \delta_{\ell\ell'}$$
$$\sum_{n_\ell=0}^{\infty} |n_\ell\rangle\langle n_{\ell'}| = 1$$

Summary

$$\hat{a}_\ell = \sqrt{\frac{2\epsilon_0 L^3 \omega_\ell}{\hbar}} \hat{\mathcal{A}}_\ell$$

Commutation relation

$$\left[\hat{a}_\ell, \hat{a}^\dagger_{\ell'}\right] = \delta_{\ell\ell'}$$

Quantized fields

$$\hat{\mathbf{E}}^{(+)} = \iota \sum_\ell \sqrt{\frac{\hbar\omega_\ell}{2\epsilon_0 L^3}} \vec{\varepsilon}_\ell \, \hat{a}_\ell \, e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}} \qquad \hat{\mathbf{B}}^{(+)} = \iota \sum_\ell \sqrt{\frac{\hbar}{2\epsilon_0 L^3 \omega_\ell}} (\mathbf{k}_\ell \times \vec{\varepsilon}_\ell) \, \hat{a}_\ell \, e^{\iota \mathbf{k}_\ell \cdot \mathbf{r}}$$

Notations

$$\mathbf{A}_\ell^{(+)}(\mathbf{r},t) = \vec{\varepsilon}_\ell \mathcal{A}_\ell(\mathbf{r}) e^{\imath(\mathbf{k}_\ell\cdot\mathbf{r}-\omega_\ell t)}$$ complex vector potential

$$\mathbf{E}_\ell^{(+)}(\mathbf{r},t) = \vec{\varepsilon}_\ell \mathcal{E}_\ell(\mathbf{r}) e^{\imath(\mathbf{k}_\ell\cdot\mathbf{r}-\omega_\ell t)}$$ complex electric field

$$= \imath\omega_\ell \vec{\varepsilon}_\ell \mathcal{A}_\ell(\mathbf{r}) e^{\imath(\mathbf{k}_\ell\mathbf{r}-\omega_\ell t)}$$

$$\mathcal{E}_\ell(\mathbf{r}) = \imath\omega_\ell \mathcal{A}_\ell(\mathbf{r})$$ electric field amplitude

$$a_\ell(\mathbf{r}) = \sqrt{\frac{2\epsilon_0 n_\ell^2 \omega_\ell L^3}{\hbar}}\mathcal{A}_\ell(\mathbf{r})$$ adimensional electromagnetic field amplitude

Three wave mixing : coupled wave equations

$$\frac{d\mathcal{E}_1}{dz} = \iota\frac{\omega_1\chi^{(2)}}{n_1 c}\mathcal{E}_2^*\mathcal{E}_3 e^{-\iota\Delta kz}$$

$$\frac{d\mathcal{E}_2}{dz} = \iota\frac{\omega_2\chi^{(2)}}{n_2 c}\mathcal{E}_1^*\mathcal{E}_3 e^{-\iota\Delta kz}$$

$$\frac{d\mathcal{E}_3}{dz} = \iota\frac{\omega_3\chi^{(2)}}{n_3 c}\mathcal{E}_1\mathcal{E}_2 e^{\iota\Delta kz}$$

Adimensionalize:

$$\frac{c}{n_1}\frac{da_1}{dz} = \gamma a_2^* a_3 e^{-\iota \Delta kz} \quad (3)$$

$$\frac{c}{n_2}\frac{da_2}{dz} = \gamma a_1^* a_3 e^{-\iota \Delta kz} \quad (4)$$

$$\frac{c}{n_3}\frac{da_3}{dz} = -\gamma a_1 a_2 e^{\iota \Delta kz} \quad (5)$$

Coupling factor γ:

$$\gamma = \chi^{(2)}\sqrt{\frac{\hbar\omega_1\omega_2\omega_3}{2\epsilon_0 L^3 n_1^2 n_2^2 n_3^2}}$$

Quantized coupled wave equations

$$a_\ell \longrightarrow \hat{a}_\ell \qquad \left[\hat{a}_\ell, \hat{a}^\dagger_{\ell'}\right] = \delta_{\ell\ell'}$$

$$\frac{c}{n_1}\frac{d\hat{a}_1}{dz} = \gamma \hat{a}^\dagger_2 \hat{a}_3 e^{-\iota \Delta k z} \tag{6}$$

$$\frac{c}{n_2}\frac{d\hat{a}_2}{dz} = \gamma \hat{a}^\dagger_1 \hat{a}_3 e^{-\iota \Delta k z} \tag{7}$$

$$\frac{c}{n_3}\frac{d\alpha_3}{dz} = -\gamma \hat{a}_1 \hat{a}_2 e^{\iota \Delta k z} \tag{8}$$

Quantized coupled wave equations

$$\frac{c}{n_1}\frac{d\hat{a}_1}{dz} = \gamma\alpha_3\hat{a}_2^\dagger \quad (9)$$

$$\frac{c}{n_2}\frac{d\hat{a}_2}{dz} = \gamma\alpha_3\hat{a}_1^\dagger \quad (10)$$

Solution

$$\hat{a}_1(z) = \hat{a}_1(0)\cosh(gz) + \hat{a}_2^\dagger(0)\sinh(gz) \quad (11)$$

$$\hat{a}_2(z) = \hat{a}_2(0)\cosh(gz) + \hat{a}_1^\dagger(0)\sinh(gz) \quad (12)$$

with $g = \frac{\sqrt{n_1 n_2}}{c}\gamma\,|\alpha_3|$

photon number at z

$$\hat{N}_1(z) = \hat{a}_1^\dagger(z)\hat{a}_1(z) \tag{13}$$

$$= \left[\hat{a}_1^\dagger(0)\cosh(gz) + \hat{a}_2(0)\sinh(gz)\right] \tag{14}$$

$$\times \left[\hat{a}_1(0)\cosh(gz) + \hat{a}_2^\dagger(0)\sinh(gz)\right] \tag{15}$$

$$= \hat{N}_1(0)\cosh^2(gz) + \left(\hat{N}_2(0) + 1\right)\sinh^2(gz) \tag{16}$$

$$+ \left[\hat{a}_1^\dagger(0)\hat{a}_2^\dagger(0) - \hat{a}_1(0)\hat{a}_2(0)\right]\cosh(gz)\sinh(gz) \tag{17}$$

photon number at z

When there is initially no photon in modes 1 and 2, the photon number in each mode after an interaction length z is:

$$\langle \hat{N}_\ell(z) \rangle = \langle 0_1 0_2 | \hat{N}_\ell(z) | 0_1 0_2 \rangle = \sinh^2(gz)$$

For low efficiencies

$$\langle \hat{N}_\ell(z) \rangle \approx g^2 z^2 = |\mathcal{E}_3|^2 \frac{\omega_1 \omega_2}{n_1 n_2 c^2} \left(\chi^{(2)} \right)^2 z^2$$

This non zero result comes from the commutation relation of the operators $\hat{a}$ and $\hat{a}^\dagger$