1 The quantized electric field

Bibliography
The Quantum Theory of Light, Loudon, Ch. 4
Introductory Quantum Optics, Gerry and Knight, Ch. 2

1.1 Introduction

We wish to find a way to express the electric and magnetic fields as quantum mechanical observables corresponding to operators $\left(\hat{E}, \hat{B}...\right)$ on a Hilbert space where the state vectors $|\psi\rangle$ describe the state of the field. Remember that in an experiment one never observes operators nor state vectors, but only expectation values or probabilities, such as $\left\langle \psi \left| \hat{E} \right| \psi \right\rangle$. The introduction of this description is an inductive leap, principally justified by its success in predicting and understanding the results of experiments.

Although there is no *ab initio* "derivation" of quantization, there are a few empirical guide posts which can help render the quantization procedure plausible. First, keeping in mind the fact that the electromagnetic field obeys a wave equation, and the idea that an electromagnetic field mode should act like a harmonic oscillator, it is natural to look to the quantum theory of the harmonic oscillator for an example of how to construct the desired operators and states. This intuition is reinforced by the observation that the quantum HO has quantized, equally spaced energy levels which might be identified with the quantized exchanges of energy postulated by Planck, and with the photons which appear to come out of treatments such as Einstein's of the interaction of fields with atoms.

1.2 The classical electric field

We will consider a field in the absence of any charges. It is convenient to introduce a vector potential $\boldsymbol{A}$ which allows us to write (using the Coulomb gauge $\boldsymbol{\nabla \cdot A = 0}$):

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t},\ \boldsymbol{B} = \nabla \times \boldsymbol{A},\ \nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = 0$$

The choice of the Coulomb gauge is arbitrary, but it results in the useful simplification that the basic equation describing electromagnetic waves depends only on the vector potential as above, while the scalar potential describes purely electrostatic effects. This situation is especially convenient in a nonrelativistic context. See Loudon ch. 4 for a good discussion of the Coulomb gauge.

Suppose the field is defined in a box of volume L^3 with periodic boundary conditions. The solutions of $\boldsymbol{A}$ can be expressed as superpositions of plane waves $\left(k = \frac{2\pi}{L}n\right)$. Let us

first consider a single mode.

$\boldsymbol{A} = \boldsymbol{\epsilon}_x \left(c e^{i(kz-\omega t)} + c^* e^{-i(kz-\omega t)} \right)$.
$\boldsymbol{E} = i\omega \boldsymbol{\epsilon}_x \left(c e^{i(kz-\omega t)} - c^* e^{-i(kz-\omega t)} \right)$
$\boldsymbol{B} = ik \boldsymbol{\epsilon_y} \left(c e^{i(kz-\omega t)} - c^* e^{-i(kz-\omega t)} \right)$
The energy in the field, averaged over an optical cycle, is:

$$H = \frac{1}{2} \int dV \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) = 2V \epsilon_0 \omega^2 c c^* \equiv \frac{1}{2} \left(\frac{p^2}{m} + m\omega^2 q^2 \right)$$

Averaging over an optical cycle eliminates terms varying as $e^{\pm 2 i \omega t}$. In the above we have introduced the "canonically conjugate variables" p and q:

$$2\omega \sqrt{V\epsilon_0} c = \omega \sqrt{m} q + i \frac{p}{\sqrt{m}}$$

$$2\omega \sqrt{V\epsilon_0} c^* = \omega \sqrt{m} q - i \frac{p}{\sqrt{m}}$$

The energy of a single mode of the field thus looks like that of a harmonic oscillator. Keep in mind that q and p are the position and momentum of a fictitious oscillator.

If it's not already obvious, we should emphasize that in the presence of multiple modes, the energy will appear as the sum of as many independent harmonic oscillators. Suppose there are two modes, 1 and 2. The cycle averaged energy is

$$\begin{aligned} H &= \epsilon_0 \int dV \left(E_1 + E_2 \right)^2 \\ &= \epsilon_0 \int dV \left(2{\omega_1}^2 c_1 {c_1}^* + 2{\omega_2}^2 c_2 {c_2}^* + \omega_1 \omega_2 \Re \left(c_1 {c_2}^* e^{i((k_1-k_2)z - (\omega_1 - \omega_2)t)} \right) \right) \end{aligned}$$

Since the modes satisfy the boundary conditions, the $e^{i(k_1-k_2)z}$ factor gives zero if integrated over the entire volume, as it must for orthogonal spatial modes. Thus the energy in this two mode situation is:

$$H = \int dV \epsilon_0 E^2 = \frac{1}{2} \left(\frac{{p_1}^2}{m_1} + m_1 {\omega_1}^2 {q_1}^2 \right) + \frac{1}{2} \left(\frac{{p_2}^2}{m_2} + m_2 {\omega_2}^2 {q_2}^2 \right).$$

This is a general property of linear wave equations: with an appropriate change of variables, the normal modes appear as independent harmonic oscillators. Thus the motivation for Rayleigh's approach (and Planck's for that matter) to compute the energy of the electromagnetic field in terms of a collection of oscillators is easily understood.

1.3 Quantizing a single mode of the field

Here is the inductive leap: let us assume that a quantum field can be described by quantum HO operators:

$q \to \hat{q}, p \to \hat{p}$, with $[q,p] = i\hbar$

We will make two more changes of variables:

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega m}}\left(\omega m\hat{q} + i\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\hat{Q} + i\hat{P}\right)$$
$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega m}}\left(\omega m\hat{q} - i\hat{p}\right) = \frac{1}{\sqrt{2}}\left(\hat{Q} - i\hat{P}\right), \quad \left[a, a^\dagger\right] = 1, \ [Q,P] = i$$

Which gives:

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \tfrac{1}{2}\right)$$
$$\hat{\boldsymbol{E}} = \boldsymbol{\epsilon}_x i\sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}e^{i(kz-\omega t)} - \hat{a}^\dagger e^{-i(kz-\omega t)}\right)$$
$$\hat{\boldsymbol{B}} = \boldsymbol{\epsilon}_y \tfrac{i}{c}\sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}}\left(\hat{a}e^{i(kz-\omega t)} - \hat{a}^\dagger e^{-i(kz-\omega t)}\right)$$

As you know, the operators a and $a^\dagger$ decrease or increase the number of excitations in the oscillator (or the field). The operators Q and P look like dimensionless momentum variables and are often referred to as "quadrature operators" for reasons that will become more clear below.

The state of a single mode field can be expressed as superpositions of the eigenstates of the Hamiltonian.

$$H|n\rangle = \left(n + \tfrac{1}{2}\right)\hbar\omega\ |n\rangle$$

Given an arbitrary state $|\psi\rangle = \sum C_n|\,n\rangle$, one can determine the expectation values of operators such as $\hat{E}$ or $\hat{\vec{N}} \propto \hat{E} \times \hat{B}$. See the exercise. *Nota bene*: the operator $p = \frac{1}{i}\sqrt{2\hbar\omega}\left(a - a^\dagger\right)$ is *not* the momentum operator for the field; it is the momentum operator for the ficticious oscillator corresponding to the field.

The operators Q and P do have a clear interpretation in terms of the electromagnetic field however. We can write the field as:

$$\hat{\boldsymbol{E}} = -\boldsymbol{\epsilon}_x\sqrt{\tfrac{\hbar\omega}{\epsilon_0 V}}\left(\hat{Q}\sin(kz-\omega t) + \hat{P}\cos(kz-\omega t)\right)$$

So that $\hat{Q}$ and $\hat{P}$ correspond to two "quadratures" of the field. The amplitude is $\sqrt{Q^2+P^2}$ and the phase is $\cos\phi = \frac{Q}{\sqrt{Q^2+P^2}}$. One can thus ask about the expectation values of the amplitude and phase of the field via these operators, and we will see later in the course how a realistic apparatus can measure them. One can also plot

them in a phase space diagram.

1.4 Examples of states of the electric field

To simplify notation, will discuss the field at the position $z = 0$, so that

$$\hat{\boldsymbol{E}} = \boldsymbol{\epsilon}_x \; i \; \sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}} \left(\hat{a}e^{-i\omega t} - \hat{a}^\dagger e^{i\omega t}\right)$$

We must keep in mind however that we sometimes must include the position dependent phase in the problem.

1.4.1 Vacuum and number states

Although the vacuum state $|n = 0\rangle$ is the ground state of the field, we have seen that it is not "empty". Its energy is $\hbar\omega/2$. We can use the field operator to find the expectation value of the field:

$$\langle 0|E|0\rangle = i\sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}} \left(\langle 0\,|\hat{a}|\,0\rangle\, e^{-i\omega t} - \left\langle 0\left|\hat{a}^\dagger\right|0\right\rangle e^{i\omega t}\right) = 0$$

Notice however that $\langle 0\,|E^2|\,0\rangle \neq 0$:

$$\langle 0\,|E^2|\,0\rangle = -\tfrac{\hbar\omega}{2\epsilon_0 V} \left\langle a^2 e^{-2i\omega t} + a^{\dagger 2} e^{2i\omega t} - aa^\dagger - a^\dagger a\right\rangle = \tfrac{\hbar\omega}{2\epsilon_0 V}$$

This means the field is a fluctuating quantity, with zero mean. Thus there is good reason for vacuum energy $\hbar\omega/2$ to be present in the Hamiltonian. The fluctuating field must contribute something to the energy. Other number states behave similarly:

$$\langle n|E|n\rangle = 0$$

$$\langle n\,|E^2|\,n\rangle = \tfrac{\hbar\omega}{\epsilon_0 V}\left(n + \tfrac{1}{2}\right).$$

The n-dependence can loosely be interpreted to mean that the "field of one photon" is $\sqrt{\frac{\hbar\omega}{\epsilon_0 V}}$. The reality of this fluctuating field can be seen in the Lamb shift and in the Casimir effect. (See Gerry and Knight, Ch. 2)

1.4.2 Coherent states

Viewed as energy eigenvectors, number states are simple, but they are non-intuitive if you want to think about electric fields. The fields we are familiar with oscillate: $E = E_0 \sin\omega t$. How can an oscillating expectation value come about? The electric field operator contains linear combinations of a and $a^\dagger$, thus it is natural to seek eigenvalues

of the annihilation operator, $\hat{a}|\,\alpha\rangle = \alpha|\alpha\rangle$. Since $\hat{a}$, is not Hermitian, its eigenvalues are complex. Also keep in mind that $\langle\alpha\,|\hat{a}^\dagger = \langle\alpha\,|\alpha^*$, but that $\hat{a}^\dagger\big|\,\alpha\big\rangle \neq \alpha^*\big|\,\alpha\big\rangle$. The expectation value for the field in such a state is

$$\langle\alpha|E|\alpha\rangle = i\,\sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}}\,(\alpha e^{-i\omega t} - \alpha^* e^{i\omega t}) = \sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}}\;\; 2|\alpha|\sin(\omega t - \phi)$$

where ϕ is related to the phase of the complex number α. It is clear that this state is the superposition of many number states, and looking at the field Hamiltonian, we are led to interpret $|\alpha|^2$ as the "typical" number of photons in the field. We can easily find this distribution and its mean:

$$\hat{a}|\,\alpha\rangle = \hat{a}\sum_{n=0}^{\infty} C_n|\,n\rangle = \sum_{n=1}^{\infty} C_n\sqrt{n}|\,n-1\rangle = \sum_{n=0}^{\infty} C_{n+1}\sqrt{n+1}\big|\,n\big\rangle$$

$$\alpha|\alpha\rangle = \alpha\sum_{n=0}^{\infty} C_n|\,n\rangle.$$

We see that:

$$|\alpha\rangle = C_0\sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}|n\rangle, \text{ and } C_0 \text{ is fixed by } \sum_{m=0}^{\infty}|\langle m|\alpha\rangle|^2 = 1$$

Result:

$$|\alpha\rangle = e^{|\alpha|^2/2}\;\sum_{n=0}^{\infty}\tfrac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

The explicit expansion of a coherent state allows us to get the photon number distribution in a coherent state:

$$P_n^{\text{coh}} = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\tfrac{|\alpha|^{2n}}{n!}\;.$$

This is the Poisson distribution. The mean photon number is $|\alpha|^2$. For $|\alpha| \gg 1$, this distribution approaches a Gaussian distribution, otherwise it is asymmetric about its mean. The Poissonian nature of the photon distribution in a coherent state recalls the statistical behavior of raindrops hitting a roof: Given a drop rate R, the mean number hitting during an interval Δt is $\bar{n} = R\Delta t$, and the distribution of hits in that interval is $P_n^{\text{rain}} = e^{-\bar{n}}\tfrac{\bar{n}^n}{n!}$. Thus, just like independent raindrops, photons in a coherent state are in some sense "independent", a statement we will make more precise later in the course.

The variance of the Poisson distribution $\Delta n^2 \equiv \overline{n^2} - \bar{n}^2$, is $\Delta n^2 = \bar{n}$. So for large $\bar{n}$ the distribution is relatively narrow. It is also interesting to calculate the variance of the electric field. One finds:

$$\big\langle\alpha\,\big|E^2\big|\,\alpha\big\rangle - \langle\alpha|E|\alpha\rangle^2 = \frac{\hbar\omega}{2\epsilon_0 V}.$$

A variance independent of α. Thus the variance of any coherent state is identical to that of the vacuum.

Are different coherent states orthogonal? By expanding the states one can show

$$\langle\beta|\alpha\rangle = \exp\left\{-\frac{|\beta|^2}{2} - \frac{|\alpha|^2}{2} + \beta^*\alpha\right\} = \exp\left\{-\tfrac{1}{2}\left(\beta\alpha^* - \beta^*\alpha\right)\right\}\exp\left\{-\tfrac{1}{2}|\beta-\alpha|^2\right)\Big\}.$$

The 1st term is a phase, $e^{i\phi}$, the second goes rapidly to zero if α and β are very different. But in general, coherent states are not orthogonal.

We have already introduced the quadrature operators Q and P. We see that:

$$\langle\alpha|Q|\alpha\rangle = \sqrt{2}\Re\,\alpha, \langle\alpha|P|\alpha\rangle = \sqrt{2}\Im\,\alpha$$

$$\Delta Q^2 = \tfrac{1}{2} = \Delta P^2.$$

We confirm the uncertainty relation $\Delta Q\Delta P > \frac{1}{2}$. The expression in terms of quadratures suggests representing the state of the field in a plane with axes $\langle Q\rangle$ and $\langle P\rangle$. A coherent state $|\alpha|e^{i\phi}$ can be represented as a vector of length $|\alpha|$ making an angle ϕ with the $\langle Q\rangle$ - axis. The fact that the values of Q and P are subject to some uncertainty is expressed by representing the tip of the vector as a circle of radius $\sim 1/2$. The vacuum is also described by such a circle, centered at the origin. For this reason one often refers to a coherent state as a "displaced vacuum". The near orthogonality (or not) of two coherent states corresponds to whether the two circles are separated or overlap.

Coherent states are so close to familiar classical fields that one can simply represent the expectation value of the field, $E(t)$ as the real part of complex number:

$$\langle E(t)\rangle \Big/ \sqrt{\tfrac{\hbar\omega}{2\epsilon_0 V}} = |\alpha|\Re e^{-i\omega t + \mathrm{i}\phi}.$$

In the complex plane, the field corresponds to point at distance $|\alpha|$ from the origin which rotates at frequency ω. The fact that the variance of the field is not zero can be accounted for by replacing the point with a Gaussian cloud of width 1/2.

1.4.3 Thermal states

Thermal radiation is characterized by the Planck distribution of energy density per unit frequency at frequency ω:

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2c^3}\frac{1}{e^{\hbar\omega/k_BT} - 1}$$

Recall that one way to arrive at this distribution is to compute the mean excitation number for the oscillators corresponding to a mode of the radiation field. In thermal equilibrium, the populations of the possible energy states, E_n of each oscillator are distributed according to the Boltzmann law: $P(E_n) \propto e^{-E_n/k_BT}$. Here P is the probability that the energy of the oscillator is E_n. Writing $E_n = n\hbar\omega$, and normalizing the series we have

$$P(n) = (1 - e^{-\hbar\omega/k_BT})e^{-n\hbar\omega/k_BT}$$

Let us now concentrate on a single mode of the field. If necessary we could isolate this mode from the others with a filter. The above formula gives us the populations of

all the eigenstates of the oscillator. Since the oscillator is in a thermal state we do not expect any coherence to exist between the different eigenstates. Because of this, there is *no statevector* that can be associated with this state and we must describe the state by a density matrix. The density matrix is of infinite dimension, but it is relatively simple: the lack of coherence means that the off-diagonal elements are zero and the diagonal element are simply the populations $P(n)$.

$$\rho = \begin{pmatrix} P(0) & 0 & \dots \\ 0 & P(1) & \\ \vdots & & \ddots \end{pmatrix}$$

The expectation value of an operator such as $\hat{E}$ is $\mathrm{Tr}(\hat{E}\hat{\rho})$ and in the exercises you will work out a few examples.

1.5 Multimode fields

For the same reasons as in the classical theory, the different modes of the electromagnetic field contribute as independent harmonic oscillators to the Hamiltonian. The procedures we have developed are thus easily generalized to a multimode field:

$\hat{H} = \Sigma_{k,s}\hbar\omega_k \left(\hat{a}_{k,s}{}^\dagger \hat{a}_{k,s} + \frac{1}{2}\right)$

$\hat{E} = \Sigma_{k,s} i\boldsymbol{\epsilon}_s \sqrt{\frac{\hbar\omega_k}{2\epsilon_0 V}} \left(\hat{a}_{k,s} e^{i(kz-\omega_k t)} - \hat{a}_{k,s}{}^\dagger e^{-i(kz-\omega_k t)}\right)$ where s identifies the 2 polarizations and $\omega_k = ck$, and all modes are orthogonal. The independence of the oscillators, is guaranteed by ensuring that the operators corresponding to different modes commute:

$\left[\hat{a}_{k,s}, \hat{a}_{k',s'}{}^\dagger\right] = \delta_{k,k'}\delta_{s,s'}$.

A multimode state can be expressed in terms of the quantum numbers of each mode:

$|\psi\rangle = |n_{k_1}, n_{k_2}, ...\rangle$ or more generally:

$|\psi\rangle = \Sigma_{n_{k_1}, n_{k_2}\dots} c_{n_{k_1}, n_{k_2}} |n_{k_1}, n_{k_2}, ...\Big\rangle$.

1.6 Interaction with an atom

An electromagnetic field in the absences of everything else is not very interesting. To observe a field, we must have it interact with some matter. Fields alter matter and vice versa and this subject is very important to us. A simple but rich model for this interaction is simply the interaction between the electric dipole of an atom and the field:

$$H_{int} = -\hat{D} \cdot \hat{E}(t)$$

The atomic dipole operator is given by $\hat{D} = q\mathbf{r}$. The field operator is the one written above. This Hamiltonian operates in a Hilbert space $|i, \psi\rangle$, in which i labels the state

of the atom and ψ the state of the field. The total Hamiltonian for a single mode field interacting with an atom is:

$$H_{atom} + \hbar\omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) - \hat{D} \cdot \hat{E}(t)$$

If we now make things even simpler and suppose that the atom only has two levels which play any role in the interaction we can discuss the energy levels (eigenstates) of the above Hamiltonian. Let us call the ground and excited states of the atom g and e respectively. In the absence of any interaction the eigenstates are $|g, n = 0\rangle$, $|e, n = 0\rangle$, $|g, n = 1\rangle$, $|e, n = 1\rangle$ etc. When, as is often the case, the field frequency is close to the energy difference of the atomic levels, the states $|e, n\rangle$, $|g, n + 1\rangle$ are nearly degenerate. If we now consider the interaction described by H_{int}, we will find new eigenstates corresponding to these two levels "repelling" each other. We can thus easily, and intuitively understand the AC Stark shift (also known as the light shift) induced on the atom by the field. This light shift is at the heart of the technique of trapping an atom in a laser beam. We can easily see that a red detuned beam lowers the energy of the atom while a blue detuned beam raises it.