NONLINEAR ELECTROMAGNETISM EXAM
M2 OPTIQUE-OMP

N. Dubreuil - I. Zaquine - F. Hache - MC Schanne-Klein

December 17, 2013

Exam without document,
Pocket calculator permitted
Duration of the exam: 3 hours
****
Please, give the question sheets back to the person who is supervising the exam.
THANK YOU.
****
This 5 pages exam contains 2 independent problems, which include questions relative to the course.

1 Three wave mixing in a waveguide configuration

This problem is divided in 4 independent parts, except for the last part 1.4 that relies on part 1.3.

1.1 Phase matching

We are considering a three wave mixing process in a waveguide.

1. What is the order of this nonlinear process?
2. How many beams incident on the waveguide are required?
3. Write the conditions that must be satisfied by:
   - the frequencies of the interacting beams
   - the wavevectors of the interacting beams
4. **Phase-matching in a waveguide, based on birefringence:** If the waveguide medium is birefringent, the phase-matching condition can be satisfied in the same way as in free space.
   - Using what you know about type I phase-matching, explain Figure 1(a).
   - Write the equation expressing the type I phase-matching condition of the interaction depicted in Figure 1(a).
5. **Modal phase-matching in a waveguide:** If the waveguide medium is not birefringent, other methods can be used, as, for instance modal phase-matching:
   - Explain Figure 1(b).
   - Write the equation expressing the type II phase-matching condition depicted in Figure 1(b).

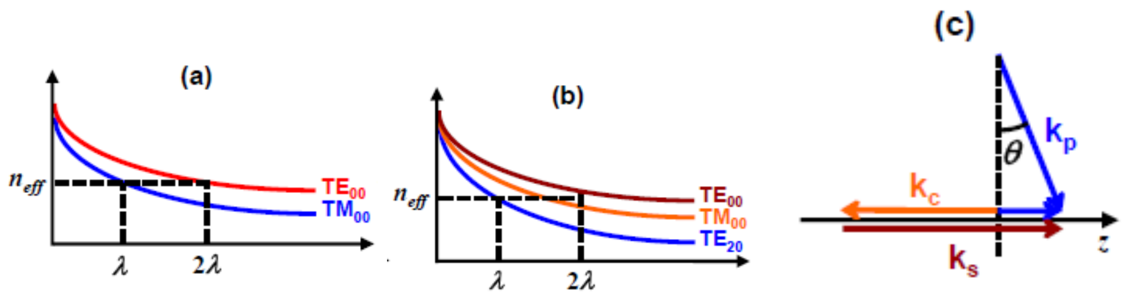

FIG. 1 – *(a) Type I phase-matching in a waveguide, based on birefringence. (b) Modal phase-matching in a waveguide. (c) Counterpropagating modal phase-matching in a waveguide.*

6. **Counterpropagating phase-matching in a waveguide :** The pump incidence is θ (see Figure 1(c)), so that only one component of the pump beam is coupled into the waveguide aligned along z : the indices s and c stand for "signal" and "complémentaire" (idler in english).
   – Explain Figure 1(c) : what is the specificity of this configuration ?
   – Write the equation expressing the type II phase-matching condition.
   – What parameter can be changed to adjust the phase-matching condition ?

1.2 Wave equation

Let us first recall the modal decomposition of the field propagating in the waveguide along the z direction :

$$\mathbf{E}(\mathbf{r},t) = \sum_{p,m} \mathbf{e}_m \phi_m^p(x,y) A_m(z) e^{-i(\omega_m t - \beta_p(\omega_m) z)}, \tag{1}$$

where the transverse profiles $\phi_m^p(x,y)$ of the modes form a complete base of orthogonal functions. In the case of a linear propagation and under the weak guidance approximation (and neglecting the anisotropy of the waveguide), the wave equation is reduced to :

$$\Delta\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = 0. \tag{2}$$

The nonlinear wave equation is simply derived from (2) by adding the nonlinear term source :

$$\Delta\mathbf{E}(\mathbf{r},\omega) + \frac{\omega^2}{c^2}\epsilon(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) = -\omega^2\mu_0\mathbf{P}^{(NL)}(\mathbf{r},\omega). \tag{3}$$

1. Derive the nonlinear wave equation of a wave at the frequency $\omega = \omega_m$ and defined by the transverse mode $\phi_m^p(x,y)$ under the slowly varying envelope approximation and explain what this approximation means.
2. In the case of three wave mixing, write the coupled equations relating the amplitudes of the three considered waves (the frequencies will be denoted ω_1, ω_2 and ω_3).
3. What is the simplification in the case of the parametric approximation ?

Let us recall that the solutions to these coupled equations in the case of the propagation of plane waves are

$$A_1(z) = \left[A_1(0)\left(\cosh\gamma z - i\tfrac{\Delta k}{2\gamma}\sinh\gamma z\right) + \tfrac{\alpha_1}{\gamma}A_2^*(0)\sinh\gamma z\right]e^{+i\frac{\Delta k}{2}z} \tag{4}$$

$$A_2(z) = \left[\tfrac{\alpha_2}{\gamma}A_1^*(0)\sinh\gamma z + A_2(0)\left(\cosh\gamma z - i\tfrac{\Delta k}{2\gamma}\sinh\gamma z\right)\right]e^{+i\frac{\Delta k}{2}z} \tag{5}$$

with

$$\begin{aligned}\alpha_1 &= \frac{i\omega_1}{n_1 c}\chi^{(2)}_{eff}A_3\\ \alpha_2 &= \frac{i\omega_2}{n_2 c}\chi^{(2)}_{eff}A_3\\ \gamma_0^2 &= \alpha_1\alpha_2^*\\ \gamma^2 &= \gamma_0^2 - \left(\frac{\Delta k}{2}\right)^2\end{aligned}$$

4. What is the physical meaning of γ_0?
5. In the waveguide configuration, one needs to substitute $\chi^{(2)}_{eff}$ for a coefficient Γ_{eff}. Assuming that the transverse profiles at ω_1 and ω_2 are equal ($\phi_1^p(x,y) = \phi_2^q(x,y)$), give an expression for Γ_{eff} from equations derived at question 2.

1.3 Spontaneous parametric down conversion efficiency

***In case you haven't completed the previous part 1.2* (especially, the expression for Γ_{eff}), proceed by assuming the plane wave solutions given by equation (4) and by considering $\Gamma_{eff} = \chi^{(2)}_{eff}$.**

The nonlinear process considered here is the spontaneous parametric down conversion.

1. What is the specificity of this nonlinear process compared to the three wave mixing process considered in the previous part?

   In order to calculate the amount of photon pairs generated at the signal ω_1, we are assuming that the signal generation originates from fluctuations in the idler ω_2, such that we consider $A_1(0) = 0$ in equation (4).
   In addition, a very weak conversion efficiency is assumed (as it is commonly observed in usual crystals) so that equation (4) must be changed according to the new assumption that $\gamma_0 \ll \Delta k$.
2. In the waveguide configuration and assuming no input at the signal frequency, show that the output power at the signal frequency ω_1 is

$$P_1(L) = \frac{\omega_1^2}{2\epsilon_0 n^3 c^3}|\Gamma_{eff}|^2 P_2(0)P_3(0)\mathrm{sinc}^2\left(\frac{\Delta k L}{2}\right)L^2, \tag{6}$$

   where the dispersion of the refractive index is neglected ($n_1 = n_2 = n_3$). The power at ω_i is given by the relation: $P_i = 2nc\epsilon_0|A_i|^2\int\int\phi_i(x,y)\phi_i^*(x,y)dxdy$.

   The generated photon pair number in the waveguide configuration can be calculated from equation (6), by considering that the initial photon number at the idler frequency is one photon per mode and the number of modes is $\Delta\omega_{PM}/(2\pi)$, with $\Delta\omega_{PM}$ the phase matching bandwidth.

3. The phase matching bandwidth coincides with a reduction of the conversion efficiency by a factor 2, i.e. for $\Delta kL \simeq \pm\pi/2$. Neglecting the dispersion of the refractive index, calculate $\Delta\omega_{PM}$.
4. Using the relation between intensity and photon number, $I = 2nc\epsilon_0|A|^2 = N\hbar\omega$, calculate the efficiency η of the spontaneous down conversion process (ratio of the signal photon number to the pump photon number)

1.4 Photon pair generation in a AlGaAs waveguide

We consider the generation of frequency degenerated photon pairs by means of spontaneous parametric down conversion of a pump beam at $\lambda_3 = 775$ nm in a AlGaAs waveguide. The effective nonlinear susceptibility of AlGaAs is $\chi^{(2)}_{eff} = 180$ pm/V. Let us assume that the transverse distribution functions at the pump and signal frequencies are equal and described by a gaussian function:

$$\phi(x,y) = e^{-\frac{r^2}{w_0^2}}, \text{ with } r^2 = x^2 + y^2.$$

1. Calculations :
   - Calculate the wavelength of the generated signal.
   - Calculate Γ_{eff} in terms of $\chi^{(2)}_{eff}$ and w_0. Note that $\iint_{-\infty}^{+\infty} e^{-\frac{r^2}{w^2}} dxdy = \pi w^2$.
   - Considering a $L = 2$ mm long waveguide with a refractive index $n = 3$, calculate $\Delta\omega_{PM}$ and the corresponding value $\Delta\lambda_{PM}$ (in nm).
   - With a pump power of 10 mW, and taking $w_0 = 1$ μm, calculate the efficiency η of the spontaneous down conversion process. Comment the value.
2. How can the type II counterpropagating modal phase-matching be used to produce a polarisation entangled state where the two photons of the pair come out at different ends of the waveguide? Write the quantum state of the produced pair.

Useful physical constants

$\hbar = 1{,}054\ 10^{-34}$ Js
$\epsilon_0 = 8.8510^{-12}$ F/m
$c = 3\ 10^8$m/s

Reference

This problem is inspired by the article : Orieux et al., "Efficient parametric generation of counterpropagating two-photon states" J. Opt. Soc. Am. B, Vol. 28, pp. 45-51 (2011)

2 Modulation instability

1. We consider an electromagnetic pulse whose electric field reads

$$\mathcal{E}(z,t) = A(z,\tau)e^{i(k_0 z-\omega_0 t)} + c.c \tag{7}$$

Time τ denotes the time in the frame moving with the pulse: $\tau = t - z/v_g$ where v_g is the group velocity.
We recall the wave equation:

$$\frac{\partial A}{\partial z} = i\frac{k_2}{2}\frac{\partial^2 A}{\partial \tau^2} + i2\varepsilon_0\omega_0 n_0 n_2 \left|A\right|^2 A. \tag{8}$$

The symbols have their usual signification; they are all supposed to be real. Explain the origin of the right-hand-side terms. Why is there not any phase matching term in this equation?

2. We suppose that the wave is a plane wave: $A(z,\tau) = A_0(z)$. Show that the modulus of A_0 remains constant during propagation. *We will suppose that this holds true in the following of this exercise.* The intensity of the wave will be denoted I_0 .

3. During the propagation, new electromagnetic waves with slightly different frequencies can appear because of noise. We want to study the consequence of this noise.
The electric amplitude now reads

$$A(z,\tau) = A_0(z) + A_1(z)e^{-i\delta\tau} + A_2(z)e^{+i\delta\tau}, \tag{9}$$

with $A_1,A_2 \ll A_0$.

(a) Which processes can generate such frequency-shifted waves?

(b) Derive the wave equations for $A_1(z)$ and $A_2(z)$ keeping only the first-order terms in A_1 and A_2 and sorting out the various terms as a function of their frequencies.

(c) Show that these equations can be written as:

$$\frac{\partial A_1}{\partial z} = i\left(2\gamma-\kappa\right)A_1 + i\gamma A_2^* e^{i2\gamma z} \tag{10}$$

$$\frac{\partial A_2^*}{\partial z} = -i\left(2\gamma-\kappa\right)A_2^* - i\gamma A_1 e^{-i2\gamma z} \tag{11}$$

Express γ and κ as a function of I_0.

(d) Eliminate A_2^* to obtain a second-order differential equation for A_1.

(e) In which circumstances can the wave A_1 experience an exponential gain during its propagation (= modulation instability)?
We suppose $k_2 > 0$, is there a threshold on I_0 for the onset of the instability?

(f) Express the gain in intensity g experienced by A_1. For which value $\delta_{\max}$ (for a fixed I_0) is the gain maximum?

(g) *Numerical application:* $n_2 = 3 \times 10^{-20}$ m^2/W (silica), $\omega_0 = 2.4 \times 10^{15}$ s^{-1}, $I_0 = 10^{15}$ W/cm^2, $k_2 = 500$ fs^2/cm. Calculate the ratio $\delta_{\max}/\omega_0$.