NONLINEAR ELECTROMAGNETISM EXAM
M2 OPTIQUE-OMP

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Exam without document,
Pocket calculator permitted
Duration of the exam: 3 hours

This 4 pages exam includes a first part with questions relative to the course, followed by 3 independent exercises. Questions should be answered in a concise and precise manner, especially those relative to the course.

1 Questions relative to the course

1. **Nonlinear scattering**
   (a) What is the physical origin of the Raman scattering process?
   (b) We consider the case of Silica material that possesses a Raman transition centered towards $\Omega_R = 13\,\text{THz}$. At room temperature, compare[1] k_BT to $\hbar\Omega_R$. Which are the consequences?
   (c) A monochromatic beam goes through a solid silica medium. The power density is high enough in order for inelastic scattering process to take place. Draw the spectrum of the scattered light. Explain the various contributions that can be observed.
2. **Stationary nonlinear wave equation in an isotropic medium**
   An electromagnetic wave is propagating in a nonlinear medium. In the general case, its complex amplitude $\mathbf{E}(\omega)$ satisfies the wave equation:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E}(\omega) = \frac{\omega^2}{c^2}\underline{\underline{\epsilon}}(\omega)\mathbf{E}(\omega) + \omega^2\mu_0\boldsymbol{\mathcal{P}}_{NL}(\omega). \tag{1}$$

   In the following, the specific case of an isotropic medium will be considered.
   (a) What can you say about $\underline{\underline{\epsilon}}(\omega)$?
   (b) Assuming a wave propagating along the z direction, rewrite Eq. (1)
   N.B. Reminder : $\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \mathbf{E} = \boldsymbol{\nabla}(\boldsymbol{\nabla}.\mathbf{E}) - \Delta\mathbf{E}$ and $\boldsymbol{\nabla}.\mathbf{D} = 0$.
   (c) Expressing the complex amplitude as $\mathbf{E} = A(z)e^{ikz}\mathbf{e}$, show that the wave equation is reduced to a first order differential equation in z. Explicit any approximation(s).

1. $k_B = 1{,}38\,10^{-23}$ F/m, $h = 6{,}62\,10^{-34}$J.s

2 Nonlinear interaction in a waveguide - Application to the case of second harmonic generation

Let us consider a waveguide, assumed lossless. The shape of the transverse dependence of its refractive index $n(\mathbf{r})$ does not change with z. In the following, $\mathbf{r}$ stands for the two-dimensional transverse vector: $\mathbf{r} = \left| \begin{array}{c} x \\ y \end{array} \right.$. A wave at frequency ω propagates in one of the propagation modes of the waveguide. Its complex amplitude can then be written:

$$\mathbf{E}(\mathbf{r},z,\omega) = \mathbf{e}\, F(\mathbf{r}) A(z) e^{i\beta z},$$

where $F(\mathbf{r})$ is the transverse distribution of the considered guided mode with a propagation constant β. $F(\mathbf{r})$ does not depend on z and is assumed to be a solution of the following equation :

$$\Delta_T F(\mathbf{r}) + \left[\frac{\omega^2}{c^2} n^2(\mathbf{r}) - \beta^2\right] F(\mathbf{r}) = 0,$$

where Δ_T is the transverse laplacien [2] and where $n^2(\mathbf{r}) = \epsilon(\mathbf{r})$.

We want to derive the nonlinear wave equation that describes the evolution of the amplitude $A(z)$ of a wave propagating in one of the guided modes described by the transverse distribution $F(\mathbf{r})$ and the propagation constant β. The power of the propagating beam is related to its amplitude A in the following way: $P = 2nc\epsilon_0 |A|^2 \iint_{-\infty}^{+\infty} |F(\mathbf{r})|^2 d^2\mathbf{r}$.

1. Using equation (1), and assuming an isotropic medium, show that in the slowly varying envelope approximation, the wave amplitude satisfies the following equation:

$$F(\mathbf{r}) 2i\beta \frac{\partial A}{\partial z} = -\omega^2 \mu_0\, \mathbf{e} \cdot \boldsymbol{\mathcal{P}}_{NL}(\mathbf{r},z,\omega) e^{-i\beta z}.$$

Important assumption : The index variation is assumed small enough to consider that $\boldsymbol{\nabla}.\mathbf{D}(\mathbf{r},\omega) = \boldsymbol{\nabla}.\left(\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r},\omega)\right) = 0$ implies $\boldsymbol{\nabla}.\mathbf{E}(\mathbf{r},\omega) \simeq 0$.

2. Then show the following relation:

$$2i\beta \frac{\partial A}{\partial z} = -\omega^2 \mu_0\, e^{-i\beta z}\, \frac{\iint_{-\infty}^{+\infty} \mathbf{e} \cdot \boldsymbol{\mathcal{P}}_{NL}(\mathbf{r},z,\omega) F^*(\mathbf{r}) d^2\mathbf{r}}{\iint_{-\infty}^{+\infty} |F(\mathbf{r})|^2 d^2\mathbf{r}}. \qquad (2)$$

Give a physical interpretation of the fraction numerator on the right hand-side of the equation. What is the denominator ?

3. Let us now consider the generation of a second harmonic wave in a waveguide with a second order nonlinearity. A pump wave at frequency ω_1 is injected in the fundamental mode of the waveguide (assumed to be single mode), which generates a beam at frequency $\omega_2 = 2\omega_1$. It will be assumed that the phase-matching condition is satisfied and that the nonlinear susceptibility of the waveguide does not depend on $\mathbf{r}$.
The complex amplitudes of the two beams are given by:

$$\mathbf{E}_1(\mathbf{r},z,\omega_1) = \mathbf{e}_1\, F_1(\mathbf{r}) A_1(z) e^{i\beta_1 z} e^{-i\omega_1 t},$$

for the pump and,

$$\mathbf{E}_2(\mathbf{r},z,\omega_2) = \mathbf{e}_2\, F_2(\mathbf{r}) A_2(z) e^{i\beta_2 z} e^{-i\omega_2 t},$$

for the second harmonic beam.

(a) Write the nonlinear polarization at frequency ω_2.

2. The total laplacian can then be written: $\Delta = \Delta_T + \frac{\partial^2}{\partial z^2}$

(b) Using equation (2), derive the nonlinear wave equation at frequency ω_2:

$$\frac{\partial A_2}{\partial z} = \cdots$$

(c) Compare this equation to the corresponding plane wave case (i.e. without taking into account any transverse mode structure). Give an expression of the ratio η between the effective nonlinear susceptibilities corresponding to these two cases, as a function of the mode transverse distributions $F_1(\mathbf{r})$ and $F_2(\mathbf{r})$.

(d) Assuming that the pump is undepleted, calculate the ω_2 beam intensity at $z = L$ (with $A_2(z = 0) = 0$).

(e) Let us assume that the transverse distribution functions of the guided modes are close to gaussian functions:

$$F_1(\mathbf{r}) = e^{-\frac{r^2}{w_1^2}} \text{ et } F_2(\mathbf{r}) = e^{-\frac{r^2}{w_2^2}}.$$

Express η as a function of w_1 and w_2. Calculate it for $w_1 = w_2$. What is the optimum ratio $\frac{w_2}{w_1}$ that gives the maximum value $\eta = 1$? Comment on this result.

Note that $\iint_{-\infty}^{+\infty} e^{-\frac{r^2}{w^2}} d^2\mathbf{r} = \pi w^2$.

(f) **Numerical example :** The waveguide is lithium niobate ($LiNbO_3$), 20 mm long, with a refractive index $n(\omega_1) = 2.155$ and $n(\omega_2) = 2.234$. In a quasi-phase matching configuration (PPLN), the effective nonlinear susceptibility (in the plane wave case) is 30 pm/V. A pulsed laser at 1064 nm, with a 1 kW peak-power, is injected in the waveguide. How much green light power is generated? Note that $w_1 = 84\,\mu$m and $w_2 = 72\,\mu$m. Conclude and comment on this result.

3 Three-photon absorption

We study the interaction between a wave at ω and a nonlinear medium with two energy levels $|a\rangle$ (lower energy state) and $|b\rangle$, such that their energy difference is equal to 3ω.

The wave at ω is characterized by an electromagnetic field mode with a wave vector $\boldsymbol{k}$, a polarization vector $\boldsymbol{e}$, and an initial photon number N. In the following, we seek to determine the variation of the photon number N that occurs during the propagation of the beam through this nonlinear material. Subsequently, we assume that the population on the level $|b\rangle$ is initially equal to zero and that one-photon or two-photon transitions are not possible.

The quantum approach followed in this exercise consists in calculating the transition rate between an initial composite state $|i\rangle$ to a final state $|f\rangle$:

$$\frac{dP_{i\to f}}{dt} = \frac{2\pi}{\hbar} |K_{fi}|^2 \rho(\Delta E),$$

where $\rho(\Delta E)$ denotes the transition lineshape of the transition $|a\rangle \to |b\rangle$, and K_{fi} is the transition rate. In the case of a three-photon transition, it takes the following form :

$$K_{fi}^{(3)} = \sum_{g,g'} \frac{\langle f|H'|g\rangle\langle g|H'|g'\rangle\langle g'|H'|i\rangle}{(E_f - E_g)(E_f - E_{g'})},$$

where g and g' denote intermediate states of the composite system, E_m (with $m = f$,g or g') the energy of the composite system in the state $|m\rangle$ and H' the interaction Hamiltonian that is given by (in the electric dipole approximation) :

$$H' = -\mathbf{E}.\mathbf{P} = -i\sqrt{\frac{\hbar}{2\epsilon_0 V}}\left[\frac{\sqrt{\omega}}{n}\mathbf{e}.\mathbf{P}\left(ae^{i\mathbf{k}.\mathbf{r}} - a^\dagger e^{-i\mathbf{k}.\mathbf{r}}\right)\right],$$

where n and V are respectively the refractive index of the material, and the quantization volume of the electromagnetic field. We remind the following relations for the annihilition and creation operators :

$$a^\dagger|N\rangle = \sqrt{N+1}|N+1\rangle \text{ et } a|N\rangle = \sqrt{N}|N-1\rangle,$$

where $|N\rangle$ is the quantum state corresponding to N photons in the electromagnetic field mode defined by $(\boldsymbol{k},\boldsymbol{e})$.

N.B.: Questions 6 to 10 can be addressed separately from the other questions.

1. Describe the initial $|i\rangle$ and the final $|f\rangle$ states of the composite system formed by the material and the electromagnetic field mode.
2. Identify the intermediate states $|g\rangle$ with a non-zero contribution to the calculation of the three-photon transition rate.
3. Give an expression for $\left|K_{fi}^{(3)}\right|^2$. Subsequently, we assume that $N \gg 1$.
4. Derive a relation for the rate of energy W that is transferred from the wave at ω to the material. We remind the relation :

$$\left(\frac{dW}{dt}\right)_\omega = \hbar\omega\frac{dP_{i\to f}}{dt}\mathcal{N}_0 V\rho_{aa}^0,$$

   with $\mathcal{N}_0$ the total density of population in the material, ρ_{aa}^0 the population in the lower level $|a\rangle$.
5. Show that the time dependence of the photon number can be written:

$$\frac{dN}{dz} = -\beta N^3, \quad (3)$$

   where β is a coefficient to be determined.

   Derivation of the nonlinear susceptibility
6. What is the order of the nonlinearity under study ?
7. In a degenerate case, derive a relation for the nonlinear polarization generated at ω, assuming that the three-photon absorption is predominant.
8. What can you tell about the real and imaginary parts of the nonlinear susceptibility ? Justify your answer.
9. Derive the nonlinear wave equation for the wave at ω.
10. Give an expression of the quantity dN/dz in terms of the nonlinear susceptibility. We remind the relation between the field amplitude and the number of photons in the mode :

$$I = 2nc\epsilon_0|A|^2 = \hbar\omega\frac{cN}{nV}.$$

11. Identifying this expression with equation (3), give an expression of the nonlinear susceptibility of the material in the degenerate case.