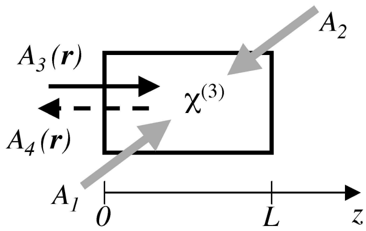
Nonlinear ELectromagnetism
Tutorial
Phase conjugate mirror using frequency degenerate four wave mixing

Let us consider a nonlinear medium with a third order susceptibility tensor $\chi^{(3)}$, in which two counter propagating waves $\boldsymbol{\mathcal{E}}_1(\boldsymbol{r},t)$ et $\boldsymbol{\mathcal{E}}_2(\boldsymbol{r},t)$, called pump waves, and a signal wave $\boldsymbol{\mathcal{E}}_3(\boldsymbol{r},t)$ interact. All these waves have the same frequency ω and the same polarisation direction $\boldsymbol{e}$. The pump waves are plane wave that propagate along the z' direction. The signal wave propagates in the z direction and no assumption is made about its spatial structure. We are interested in a nonlinear polarisation, generated through the interaction of these waves with the $\chi^{(3)}$ nonlinear medium, that will radiate a field at frequency ω, propagating in the $-z$ direction and that is phase conjugate of the field $\boldsymbol{\mathcal{E}}_3$.



The fields can be written in the following way:

$$\begin{aligned}
\boldsymbol{\mathcal{E}}_1(\boldsymbol{r},t) &= \boldsymbol{e}A_1(x,y,z)e^{i(kz'-\omega t)} + C.C &= \boldsymbol{E}_1 e^{-i\omega t} + C.C \\
\boldsymbol{\mathcal{E}}_2(\boldsymbol{r},t) &= \boldsymbol{e}A_2(x,y,z)e^{-i(kz'+\omega t)} + C.C &= \boldsymbol{E}_2 e^{-i\omega t} + C.C \\
\boldsymbol{\mathcal{E}}_3(\boldsymbol{r},t) &= \boldsymbol{e}A_3(x,y,z)e^{i(kz-\omega t)} + C.C &= \boldsymbol{E}_3 e^{-i\omega t} + C.C
\end{aligned}$$

1. The nonlinear wave equation in an isotropic medium is:

$$\Delta \boldsymbol{\mathcal{E}} - \frac{1}{c^2}\frac{\partial^2(\epsilon \boldsymbol{\mathcal{E}})}{\partial t^2} = \mu_0 \frac{\partial^2 \boldsymbol{\mathcal{P}}_{NL}}{\partial t^2}.$$

   An electromagnetic field propagating in the $+z$ direction is considered:

$$\boldsymbol{\mathcal{E}} = \boldsymbol{e}A(x,y,z)e^{i(kz-\omega t)} + C.C.$$

   The nonlinear polarisation is:

$$\boldsymbol{\mathcal{P}}_{NL} = \boldsymbol{P}_{NL} e^{i(\boldsymbol{k}_p.\boldsymbol{r}-\omega t)} + C.C.$$

   Write the nonlinear wave equation for the field amplitude $A(x,y,z)$. Write it then in the case of an electromagnetic field propagating in the $-z$ direction.

2. From the general expression of the nonlinear polarisation, only keep the terms radiating a field with an amplitude that is proportional to $\boldsymbol{E}_3^*$.

3. For which term (denoted $\boldsymbol{\mathcal{E}}_4$) is the phase-matching condition with the phase conjugate field $\boldsymbol{\mathcal{E}}_3$ automatically satisfied?

4. Assuming the parametric regime (non depletion of the pump waves), write the coupled equation system for the amplitudes $A_3(x,y,z)$ and $A_4(x,y,z)$ of the signal and conjugate waves.

5. Write this equation system in the Fourier space using:

$$A(x,y,z) = \iint_{-\infty}^{+\infty} F(k_x,k_y,z)e^{i(k_x x+k_y y)}dk_x dk_y.$$

6. Solve this system with the boundary conditions defined in the direct space: $A_4(z = L) = 0$ et $A_3(z = 0) = A_3(0)$, where L is the nonlinear medium thickness

7. Prove then that the amplitude $A_4(x,y,z = 0)$ is proportional to $A_3^*(x,y,z = 0)$.

8. Plot the general shape of the intensities of the signal and conjugate waves in the nonlinear medium. Comment on this result.