Tutorial n°9 : Stokes - Anti-Stokes coupling in Stimulated Raman Scattering

Solutions of the coupled equations :

$$\begin{cases} \dfrac{dA_S}{dz} = g_S A_S - \dfrac{\omega_S}{2n_S c} \chi_{eff}^{(3)} A_{AS}^* A_L^2 e^{i\Delta k z} \\[2ex] \dfrac{dA_{AS}^*}{dz} = -g_{AS} A_{AS}^* + \dfrac{\omega_{AS}}{2n_{AS} c} \chi_{eff}^{(3)} A_S A_L^{*} e^{-i\Delta k z} \end{cases} \quad (1)$$

with $\Delta k = 2k_L - k_{AS} - k_S$

Solution for $A_S(z)$, seeking $A_L(z) = c^{ste}$ (undepleted pump)

$\Rightarrow$ derivation of (1) $\Rightarrow \dfrac{d^2 A_S}{dz^2}$ ...

$$\hookrightarrow \boxed{\frac{d^2 A_S}{dz^2} + (g_{AS} - g_S - i\Delta k)\frac{dA_S}{dz} + i\Delta k\, g_S A_S = 0} \quad (2)$$

Seeking solutions of (2) on the form:

$$\boxed{A_S(z) = a\, e^{r_1 z} + b\, e^{r_2 z}}$$

with r_1, r_2 solutions of the eq.: $r^2 + (g_{AS} - g_S - i\Delta k) r + i\Delta k g_S = 0$ (3)

Discriminant of (3) : $\Delta = \rho e^{i\varphi}$

with $\boxed{\rho^2 = \left[(g_{AS} - g_S)^2 - \Delta k^2\right]^2 + 4\Delta k^2 (g_{AS} + g_S)^2}$

and
$$\begin{cases} \varphi = \text{Arctan}\, \dfrac{2\Delta k (g_{AS} + g_S)}{\Delta k^2 - (g_{AS} - g_S)^2} & \text{if } g_{AS} - g_S > \Delta k \\[2ex] \text{or} \quad \varphi = \pi + \text{Arctan}\, \dfrac{2\Delta k (g_{AS} + g_S)}{\Delta k^2 - (g_{AS} - g_S)^2} & \text{if } g_{AS} - g_S < \Delta k \end{cases}$$

Boundary conditions:

$$A_S(0) = c^{\underline{\Delta G}} \quad \text{and} \quad \left(\frac{dA_S}{dz}\right|_{z=0} = g_S A_S(0)$$

$$\Rightarrow \frac{A_S(z)}{A_S(0)} = \frac{g_S - r_2}{r_1 - r_2} e^{r_1 z} + \frac{r_1 - g_S}{r_1 - r_2} e^{r_2 z}$$

with
$$\begin{cases} r_1 = \frac{1}{2}\left[g_S - g_{AS} + i\Delta k + \sqrt{\rho}\left(\cos\frac{\psi}{2} + i\sin\psi/2\right)\right] \\ r_2 = \frac{1}{2}\left[g_S - g_{AS} + i\Delta k - \sqrt{\rho}\left(\cos\psi/2 + i\sin\psi/2\right)\right] \end{cases}$$

Determination of $\frac{I_S(z)}{I_S(0)} = \frac{|A_S(z)|^2}{|A_S(0)|^2}$??

$$\frac{A_S(z)}{A_S(0)} = \frac{e^{-i\psi/2}}{\sqrt{\rho}}\left[(g_S - r_2)e^{r_1 z} + (r_1 - g_S)e^{r_2 z}\right]$$

we write: $r_1 = \frac{1}{2}(a_1 + ib_1)$ and $r_2 = \frac{1}{2}(a_2 + ib_2)$

$$\frac{I_S(z)}{I_S(0)} = \frac{1}{4\rho}\left[A e^{a_1 z} + B e^{a_2 z} + C e^{\frac{a_1 + a_2}{2} z}\right]$$

Finally!!

$$\boxed{\frac{I_S(z)}{I_S(0)} = \frac{1}{4\rho}\left[A e^{(g_S - g_{AS} + \sqrt{\rho}\cos\psi/2) z} + B e^{(g_S - g_{AS} - \sqrt{\rho}\cos\psi/2) z} + 2e^{(g_S - g_{AS}) z}(\alpha\cos\delta z + \beta\sin\delta z)\right]}$$

with:
$$\begin{cases} A = (2g_S - a_2)^2 + b_2^2 = (g_S + g_{AS} + \sqrt{\rho}\cos\psi/2)^2 + (\Delta k - \sqrt{\rho}\sin\psi/2)^2 \\ B = (a_1 - 2g_S)^2 + b_1^2 = (g_S + g_{AS} - \sqrt{\rho}\cos\psi/2)^2 + (\Delta k + \sqrt{\rho}\sin\psi/2)^2 \end{cases}$$

$$\begin{cases} \delta = \sqrt{\rho}\,\sin\psi/2 \\ \alpha = \rho - (g_S + g_{AS})^2 - \Delta k^2 \\ \beta = 2\sqrt{\rho}\,[\Delta k \cos\psi/2 + (g_S + g_{AS}) \sin\psi/2 \,. \end{cases}$$

<u>CONCLUSSION</u>: We ~~consid~~ consider two "limit situations"

$\boxed{1^{st}\text{ case}}$ $\Delta k \gg g_{AS} - g_S$ = strong phase-mismatch

$$\hookrightarrow \rho \simeq \Delta k^2 \left[1 + \frac{2(g_{AS}+g_S)^2 - (g_{AS}-g_S)^2}{\Delta k^2} \right]$$

$$\Rightarrow \rho \simeq \Delta k^2$$

and $\psi \simeq \pi + \operatorname{Arctan} \dfrac{2(g_{AS}+g_S)}{\Delta k}$

$$\psi/2 \simeq \frac{\pi}{2} + \frac{g_{AS}+g_S}{\Delta k}$$

$$\hookrightarrow g_S - g_{AS} + \sqrt{\rho}\cos\psi/2 \simeq -2g_{AS}$$

$$\rightarrow \frac{B}{4\rho} \simeq 1 \text{ and } A \simeq 0$$

$$\frac{\alpha}{4\rho} \simeq \frac{g_{AS}\,g_S}{\Delta k^2}$$

$$\frac{\beta}{4\rho} \simeq \frac{(g_S+g_{AS})^2}{3\Delta k^3}$$

and $\delta \simeq \Delta k$

$$\Rightarrow \frac{I_S(z)}{I_S(0)} = e^{2g_S z} + 2e^{(g_S-g_{AS})z}\left[\frac{g_{AS}\,g_S}{\Delta k^2}\cos\Delta k z + \frac{(g_S+g_{AS})^2}{3\Delta k^3}\sin\Delta k z\right]$$

- Assuming a long interaction length ($z \gg 1$)
- a low dispersion medium. $n_S \simeq n_{AS}$
- Since $\omega_S < \omega_{AS} \Rightarrow g_S > g_{AS}$

$$\boxed{\frac{I_S(z)}{I_S(0)} \simeq e^{2g_S z}}$$

$\Rightarrow$ Similar expression to the SRS situation (without coupling between S and AS)

$\boxed{2^{nd} \text{ case}}$ $\Delta k \ll g_{AS} - g_S$

Phase-matching condition fullfiled

For a long interaction length:

$$\boxed{\frac{I_S(z)}{I_S(0)} = e^{\Gamma z}} \quad \text{with} \quad \Gamma = \frac{2 g_{AS}\, g_S\, \Delta k^2}{(g_{AS} - g_S)^2} \ll 2 g_S$$

lower amplification in presence of the coupling between Stokes and Anti-Stokes waves $\Rightarrow$ Condition: $\Delta k \ll g_{AS} - g_S$