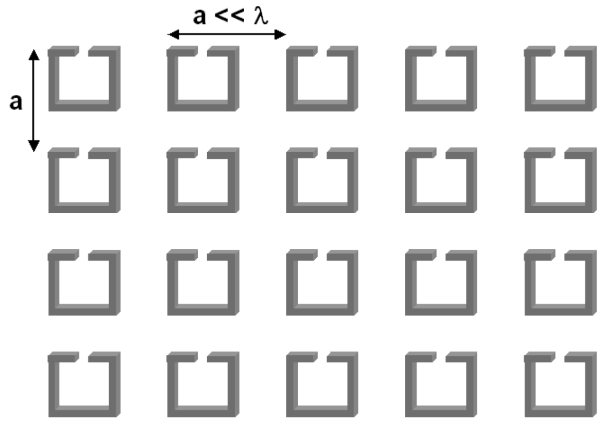
Metamaterials with artificial magnetism (negative μ)

The objective is to study with a simple model the appearance of artificial magnetism (effective permeability different from unity) in metamaterials that are made of non-magnetic materials. We consider a metamaterial made of spit-ring resonators (SRRs), i.e., metallic rings that have been cut. The SRRs form a cubic array with a period a that is much smaller than the wavelength, $a << \lambda$.



Metamaterial made of a cubic array of metallic split-ring resonators (SRRs).

1. <u>Relative permittivity and relative permeability</u>

We first remind the definitions of the relative permittivity ε_r and the relative permeability μ_r of a material. In the linear regime, the relative permittivity is defined from the ratio between the electric displacement **D** and the incident electric field **E** that has induced this displacement, $\mathbf{D} = \varepsilon_0\varepsilon_r\mathbf{E}$. The relative permittivity is different from unity because of the presence of charges in the material; when illuminated by the electric field **E**, the charges move and create a polarization **P** in the medium,

$$\mathbf{P} = \varepsilon_0\chi_e\mathbf{E} \quad \text{and} \quad \mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P},$$

with χ_e the linear electric susceptibility of the material. In general, for anisotropic media, ε_r and χ_e are 3x3 tensors.

(a) From the relations between **E**, **D** and **P**, show that $\varepsilon_r = 1 + \chi_e$.

(b) The macroscopic polarization **P** can be expressed as the contribution of all the elementary dipoles $\mathbf{p} = \alpha_e\mathbf{E}$ induced in the material, $\mathbf{P} = N\mathbf{p}$, where N is the number of dipoles per unit volume (the number of dipoles inside one unit cell of the crystal divided by the volume of the unit cell) and α_e is the electric polarisability of the dipoles. The relative permittivity can then be expressed as a function of the polarisability,

$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_0}. \qquad (1)$$

(c) Similarly, the relative permeability can be derived as a function of the magnetic polarisability α_m of the elementary magnetic dipoles induced in the material by the incident magnetic field, $\mathbf{m} = \alpha_m\mathbf{H}$. By changing **E**, **D** and **P** into **H**, **B** and **M**, show that

$$\mu_r = 1 + \frac{N\alpha_m}{\mu_0}. \tag{2}$$

For simplicity, the previous derivations have skipped an important issue in the definition of the material parameters ε_r and μ_r: local field effects have to be taken into account. Indeed, the induced dipole moment at **r** depends on the total electric field at **r**, which is different from the incident electric field because of the contribution of all the other dipoles in the material,

$$\mathbf{p} = \alpha_e \mathbf{E}_{local} \quad \text{with} \quad \mathbf{E}_{local} = \mathbf{E} + \mathbf{E}_{dipoles},$$

where the local electric field $\mathbf{E}_{local}$ is the sum of the incident electric field **E** and the electric field $\mathbf{E}_{dipoles}$ that is radiated by all the dipoles around. The usual way to take the local field effect into account [see for instance J.D. Jackson, *Classical Electrodynamics*, Section 4.5 (John Wiley, New York, 1999)] leads to the Clausius-Mossotti equation relating the polarisability to the permittivity,

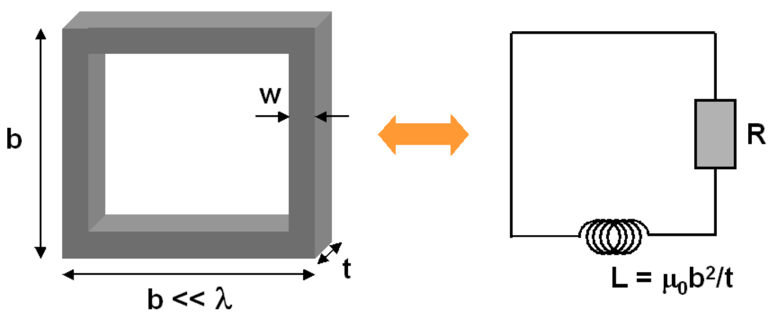
$$\varepsilon_r = 1 + \frac{N\alpha_e}{\varepsilon_0} \frac{1}{1 - N\alpha_e/(3\varepsilon_0)}. \tag{3}$$

Equation (3) shows that local field effects can be taken into account by introducing an additional factor in Eq. (1). The same procedure can be applied to the relative permeability. Hereafter, for the sake of simplicity, we do not introduce this additional term and we use Eqs. (1) and (2), but the following derivations can be easily done with Eq. (3).

Equation (2) evidences that the realization of an effective permeability different from unity requires the presence of magnetic dipoles with a size much smaller than the wavelength. If the elementary electric dipole is simply a charge q in motion, $\mathbf{p} = q\mathbf{r}$, the elementary magnetic dipole is an infinitesimal loop of section S in which a current I flows, $\mathbf{m} = \mu_0 SI\mathbf{u}$, with **u** a unitary vector perpendicular to the plane of the loop. Hereafter, we calculate the effective permeability of a metamaterial made of two different types of current loops, a closed metallic ring and a split ring.

2. Effective permeability of a metamaterial made of closed metallic rings

We first consider a metamaterial made of a cubic array (period a << λ) of closed metallic rings. The rings are square with a size b << λ. The metallic wire composing the rings have a rectangular section given by the product wt. We apply a magnetic field **H** and we want to determine the response of the metamaterial in the form of its effective permeability by using Eq. (2). The magnetic field is perpendicular to the plane of the ring and it induces a magnetic momemt **m** that is also perpendicular to the plane. Therefore we work hereafter with scalar quantities that are the z-components of the vectors.



Analogy between a closed metallic ring with a size much smaller than the wavelength and a RL circuit.

Since the size of the ring is much smaller than the wavelength, we are in the quasi-static regime (retardation effects can be neglected) and the ring can be modeled by a RL circuit, an inductance L and a resistance R that accounts for ohmic losses in the metal. If we assume that the metallic ring is equivalent to one loop of a solenoid, then we can express the inductance as a function of the geometric parameters, $L = \mu_0 b^2/t$.

We recall an usual relationship for electrical engineers working with currents induced by time-varying magnetic fields, namely Faraday's law that relates the temporal derivative of Φ, the magnetic flux through the section of the loop, to the induced voltage U,

$$U = -d\Phi/dt = i\omega\Phi. \tag{4}$$

Note that we are working with the convention $\exp(-i\omega t)$ for the time-dependence of the harmonic fields, which is not the usual convention in electrical engineering.

(a) By using Faraday's law and Ohm's law, give the expression of the current I induced in the ring by the incident magnetic field. Since $b << \lambda$, we can assume that the magnetic field H is constant over the area of the ring. From this expression, show that the magnetic polarisability α_m of one metallic ring is given by

$$\alpha_m = i\omega\mu_0^2 S^2/Z, \tag{5}$$

with $Z = R - i\omega L$ the total impedance of the ring and $S = b^2$ the ring section.

(b) From Eq. (2), we can deduce that the effective permeability of the closed ring array is given by

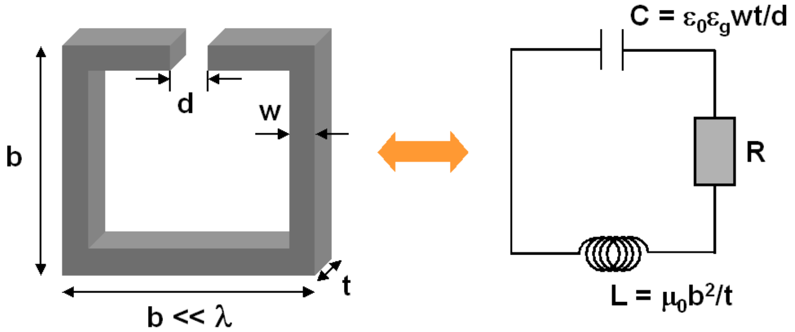
$$\mu_r = 1 - \frac{F}{1 + i/Q}, \tag{6}$$

with $Q = \omega\tau = \omega L/R$ the quality factor of the ring and $F = \mu_0 N S^2/L$ (N is the number of rings per unit volume in the cubic array). What is the physical meaning of the parameter F ?

(c) Give the expression of the effective permeability in the limit of small losses, i.e., $Q >> 1$. Since $F < 1$, can the real part of the permeability become negative ? Would you say that closed metallic rings have a small or large impact on the effective permeability ?

3. Effective permeability of a metamaterial made of metallic split-ring resonators

We now consider a metamaterial made of a cubic array (period $a << \lambda$) of split rings. A gap of size d has been cut into the metallic rings studied in previous Section.



Analogy between a split metallic ring with a size much smaller than the wavelength and a RLC circuit.

The split ring can be modeled by a RLC circuit. From the formula of a planar capacitor, we can deduce an approximate expression for the capacitance $C = \varepsilon_0\varepsilon_g wt/d$, with ε_g the relative permittivity inside the gap.

(a) What has to be changed in Eq. (5) to get the magnetic polarisability α_m of the split ring ?

(b) Show that the effective permeability of the split ring array is given by

$$\mu_r = 1 - \frac{F}{1 - \omega_0^2/\omega^2 + i/Q} = (1-F)\frac{\omega^2 - \omega_F^2 + i\omega^2/Q_F}{\omega^2 - \omega_0^2 + i\omega^2/Q}, \qquad (7)$$

with $\omega_0^2 = 1/(LC)$ the resonance frequency of the split ring, $Q = \omega\tau = \omega L/R$ the quality factor, $\omega_F^2 = \omega_0^2/(1 - F)$ and $Q_F = Q(1 - F)$.

(c) Because of the introduction of a small gap in the rings (the capacitance), the structure is now resonant. In order for the resonance wavelength to be in the spectral range under study, $\lambda_0 >> b$, which condition should be met by the gap size ?

(d) In the ideal case of a lossless system ($1/Q = 0$), what is the frequency range over which the effective permeability can be negative ? For a given parameter F, what is the main impact of the losses ?