Quantization of a harmonic oscillator by Dirac's method

Consider a one dimensional harmonic oscillator of mass m and frequency ω described by the position q and momentum p. The hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 \ . \tag{1}$$

Let's introduce the dimensionless variables $Q = \sqrt{\frac{m\omega}{\hbar}}\, q$ and $P = \frac{p}{\sqrt{m\hbar\omega}}$. Then

$$H = \frac{\hbar\omega}{2} \left(P^2 + Q^2\right) \ . \tag{2}$$

To quantize the oscillator, one replaces the classical variables p and q by the operators $\hat{p}$ and $\hat{q}$, and one imposes the commutation rules for conjugate variables: $[\hat{q}, \hat{p}] = i\hbar$, which means that $[\hat{Q}, \hat{P}] = i$.

An elegant way to find the spectrum and eigenstates of H is to introduce the operators

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{Q} + i\hat{P}) \text{ and } \hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{Q} - i\hat{P}) \ . \tag{3}$$

With this definition $H = \frac{\hbar\omega}{2}(\hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a})$. The operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is hermitian and therefore has eigenvectors (which we shall call $|n\rangle$) with real eigenvalues n. We want to show that n is a non-negative integer and find the eigenfunctions in position space $\phi_n(q)$.

1. Show that $n \geq 0$ by considering the modulus of the vector $\hat{a}|n\rangle$: $||\hat{a}|n\rangle||^2$.

2. Demonstrate the following commutation relations: $[\hat{a}, \hat{a}^\dagger] = 1$, $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$

3. Use the commutation relations to show that $\hat{N}\hat{a}|n\rangle = (n-1)\hat{a}|n\rangle$ and $\hat{N}\hat{a}^\dagger|n\rangle = (n+1)\hat{a}^\dagger|n\rangle$. This result is the origin of the names "annihilation" and "creation" operators.

4. These equations mean that $\hat{a}|n\rangle = c_-|n-1\rangle$ and $\hat{a}^\dagger|n\rangle = c_+|n+1\rangle$. We choose c_+ and c_- real. Demonstrate that $c_+ = \sqrt{n+1}$ and $c_- = \sqrt{n}$.

5. Show that n has to be an integer (hint: assume n non integer and get a contradiction).

6. Recall that $\hat{p} = -i\hbar\frac{d}{dq}$ to find the ground-state wavefunction $\phi_0(q)$ (hint: express the relation $\hat{a}|0\rangle = 0$ in position space).

7. Explain how you get all the $\phi_n(q)$.