Quantum Optics exam

M2 LOM and Nanophysique

27 November 2018

Allowed documents: lecture notes and problem sets. Calculators allowed.
Aux francophones (et francographes) : vous pouvez répondre en français.

1 Phase shifts in the Hong Ou Mandel effect

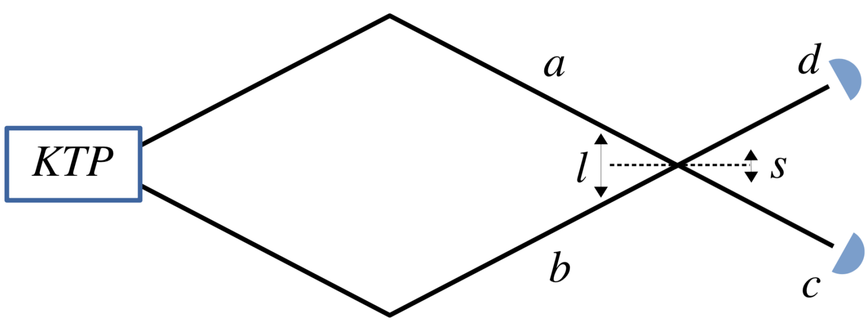

Figure 1: *The Hong Ou Mandel (HOM) experiment. A 50/50 beam splitter, shown as a dotted horizontal line, couples two input ports, a and b to two output ports c and d. The experiment consists in moving the beam splitter by a distance l, and observing the concident count rate between the detectors at c and d as a function of l. We also consider an additional, random displacement of the beam splitter s.*

In the HOM experiment, Fig. 1, the beam splitter position l is displaced over a few tens of μm to trace out a "dip" in the coincidence rate. On the other hand, the wavelength λ of the light involved in the interference is much smaller, about 700 nm, and the experimenters evidently make no effort to control the position of the beam splitter to much better than this wavelength. Presumably, the position of the mirror fluctuates on distances of order 1 μm. Call this fluctuating distance s, and define a phase $\phi/2 = 2\pi s/\lambda$. We will describe the effect of a variation in s on the input modes a and b by the replacements $\hat{a} \to \hat{a}e^{i\phi/2}$ and $\hat{b} \to \hat{b}e^{-i\phi/2}$. In the following you can neglect the multimode nature of the HOM effect and treat the problem with only 2 input modes a, b and 2 output modes c, d.

1. The above treatment of s neglects the effect on the overlap of the light pulses. Why is this neglect reasonable?

2. How do $w_c^{(1)}$ and $w_d^{(1)}$ depend on ϕ in the HOM experiment?

3. How does the coincidence rate $w_{c,d}^{(2)}$ depend on ϕ? Was it reasonable for HOM to ignore the stability of their beam splitter?

Now suppose that the state consists of identical coherent states in each input: $|\alpha, \alpha\rangle$. For simplicity you can assume that α is a real number.

4. How do $w_c^{(1)}$ and $w_d^{(1)}$ depend on ϕ ?

5. How does the coincidence rate $w_{c,d}^{(2)}$ depend on ϕ?

6. What is the average coincidence rate if the phase ϕ fluctuates from one pulse to the next?

7. If you do the HOM experiment using coherent states in the inputs and with a fluctuating mirror, will you observe a dip?

2 Generation of various quantum states of a trapped ion

In this problem, we consider an ion with internal ground state $|g\rangle$ and excited state $|e\rangle$, separated by a transition at frequency ω_0. The ion is trapped in a harmonic potential using a combination of electric fields. We will assume that the motion of the ion is one dimensional along an axis x, with an oscillation frequency Ω (careful: in this problem Ω will not be a Rabi frequency, but the oscillation frequency of the ion in the trap!). We will consider the motion of the ion as quantum, with quanta of motion $|n\rangle$. A laser field interacts with the ion, with a frequency ω close to ω_0 and a wavevector k. The laser field will be treated *classically*. The electric field of the laser is polarized along the z axis, and is $E(x,t) = \frac{1}{2}E_0(e^{i(kx-\omega t)} + e^{-i(kx-\omega t)})$. The dipole matrix element of the transition between $|g\rangle$ and $|e\rangle$ is $d = \langle e|\hat{D}|g\rangle$ and the Rabi frequency is $g = -dE_0/(2\hbar)$.

1. Write the Hamiltonian H_{vib} of the motion of the ion in the trap. Call a and $a^\dagger$ the annihilation and creation operators associated with the motion. Give the eigenenergies as a function of n and Ω.

2. The interaction Hamiltonian between the ion and the laser is $H_{\text{int}} = -\hat{D}E(x,t)$, with $\hat{D} = d(\sigma_+ + \sigma_-)$, where $\sigma_+ = |e\rangle\langle g|$ and $\sigma_- = |g\rangle\langle e|$. Unlike the lectures, we will not assume that the size of the ion wavepacket is smaller than the wavelength of the transition. This means that the position x is an operator in the expression of the electric field: $\hat{x} = \Delta x(a + a^\dagger)$, with $\Delta x = \sqrt{\frac{\hbar}{2m\Omega}}$. Expanding H_{int} in powers of $\eta = k\Delta x$ up to first order in η, and neglecting non-resonant terms gives:

$$\begin{aligned} H_{\text{int}} &\approx \hbar g(\sigma_+ e^{-i\omega t} + \sigma_- e^{i\omega t}) \\ &+ i\hbar g\eta(\sigma_+ a\, e^{-i\omega t} - \sigma_- a^\dagger e^{i\omega t}) \\ &+ i\hbar g\eta(\sigma_+ a^\dagger e^{-i\omega t} - \sigma_- a\, e^{i\omega t}) + \mathcal{O}(\eta^2) \; , \end{aligned} \tag{1}$$

Write the full Hamiltonian including the internal states of the atoms (assume $|g\rangle$ has energy 0) and the external degree of freedom. Show that this Hamiltonian resembles the Jaynes-Cummings Hamiltonian. What are the differences with respect to the Jaynes-Cummings model? What is the equivalent of the cavity here?

3. The Hilbert space describing the ion is spanned by the basis $\{|g\rangle, |e\rangle\} \otimes \{|n\rangle\}$. Above, we wrote the interaction Hamiltonian as the sum of three terms H_1, H_2 and H_3 corresponding to the three lines in Eq.(1). What are the states coupled by each of these terms? What are the associated matrix elements as a function of g, η and n?

4. Draw the energy levels of the ion including the internal and external degrees of freedom, remembering that $\omega_0 \gg \Omega$. Starting from a state $|g, n\rangle$, what are the frequencies of the laser necessary to excite the system?

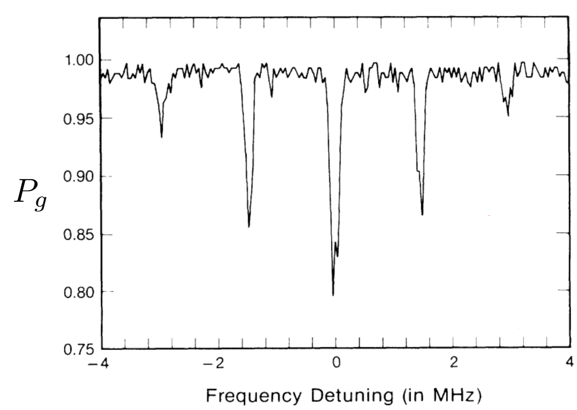

Figure 2: *Spectroscopy of a Be^+ ion in a trap. Probability to measure the ion in state $|g\rangle$ after sending a laser onto it, as a function of the frequency detuning $\Delta = \omega - \omega_0$.*

5. Figure 2 shows an example of a spectroscopic measurement on a single Be^+ ion confined in a trap. Interpret the figure and give the oscillation frequency Ω of the ion in the trap.

6. Calculate numerically value of η for this ion ($\lambda = 313\,\text{nm}$, $m = 1.5 \times 10^{-26}\,\text{kg}$).

In the remainder of this problem, we will assume that the laser frequency is $\omega = \omega_0$. Thus the part of the interaction Hamiltonian proportional to η can be ignored.

7. **Preparation of a coherent state of the motion**. It is possible experimentally to prepare the ion in the vibrational ground state $|n = 0\rangle$, while being in its internal state $|g\rangle$. Starting from this state, one displaces suddenly the center of the harmonic trap by an amount x_0. The new state of motion of the ion is $|\psi_{\text{vib}}\rangle = \exp(i\hat{p}x_0/\hbar)|n = 0\rangle$, with $\hat{p} = i\sqrt{\frac{m\hbar\Omega}{2}}(a - a^\dagger)$. Use the Campbell-Baker-Hausdorff theorem

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \tag{2}$$

   to show that $|\psi_{\text{vib}}\rangle$ is a coherent state with an amplitude α expressed as a function of x_0, $\hbar$, Ω and m.

8. If the ion is prepared in the state $|\psi(t = 0)\rangle = |\alpha\rangle$, show that $|\psi(t)\rangle = e^{-i\frac{\Omega}{2}t}|\alpha e^{-i\Omega t}\rangle$.

9. Calculate the average position $\langle\hat{x}\rangle(t)$ of the ion in the trap as a function of x_0 and Ω, as well as the variance on the position. Interpret.

10. **Generation of a Schrödinger cat of the atomic motion.** We want to generate a state of the ion motion of the form $(|\alpha\rangle \pm |-\alpha\rangle)\sqrt{2}$. What is the meaning of this state in terms of classical motion?

11. To generate such states, start with the ion prepared in $|g, n = 0\rangle$ and apply a pulse $\pi/2$ at the frequency $\omega = \omega_0$, thus acting on the internal state only. Recall that a *resonant* Rabi oscillation between any two states $|a\rangle$ and $|b\rangle$ coupled with a Rabi frequency g is described by the unitary matrix

$$\hat{R}(t) = \begin{pmatrix} \cos\frac{gt}{2} & i\sin\frac{gt}{2} \\ i\sin\frac{gt}{2} & \cos\frac{gt}{2} \end{pmatrix}_{|a\rangle,|b\rangle} . \tag{3}$$

   What is the state of the ion after the pulse?

12. Using an extra laser beam it is possible to shift the position of the ion in the trap by an amount x_0 *only* when the ion is in state $|e\rangle$. Using question 8, what is the resulting state (including internal and external degrees of freedom)? Call α the amplitude of the coherent state of motion.

13. Before the ion has the time to move (i.e. on a time scale short compared to $1/\Omega$) we apply a π-pulse such that $gt = \pi$ with $\omega = \omega_0$. What is the resulting state?

14. We then apply the laser that shifts the position of the ion when it is in $|e\rangle$, but this time shifting the position by $-x_0$. What is the state now?

15. To complete the sequence, one applies a last $\pi/2$-pulse with a frequency $\omega = \omega_0$. Show that the final state involves 2 possible Schrödinger cat states depending on the internal state of the ion.

16. Explain how you conditionally prepare each of these cat states.

3 Article Vest *et al.*, "Anti-coalescence of bosons on a lossy beam splitter"

The introduction to this paper contains a discussion of how one can modify the spatial symmetry of the state of two photons by simultaneously modifying their polarizations. Note however, that unlike photons, surface plasmon polaritons can only have one polarization. Therefore, you should keep in mind that this symmetry modification idea is included to put the experiment into a more general context and that states such as $(|H\rangle_1|V\rangle_2 + |V\rangle_1|H\rangle_2)$ do not play a role in the experiment.

1. What is the spectral width of the plasmons used in the experiment?

2. Give a possible reason for the dip not going to zero in Fig. 3.

3. The authors speak of (possibly complex) transmission and reflection coefficients t and r. In our notation, they correspond to a matrix connecting the input modes to the output modes:
$$\begin{pmatrix} t & r \\ r & t \end{pmatrix} \tag{4}$$
If t and r are real, this matrix is *not* equivalent to the one we used in the lectures to describe the action of a beam splitter. Under what circumstances is the authors' formulation valid?

4. Show that, if t and r are real, one expects the count rates in the individual detectors to behave as in the inset to Fig. 4 of the paper.

5. Is it possible to model the beam splitter in this experiment by a lossless beam splitter with 50% attenuators at each input?