Physical Optics
exercises on chapters 1, 3, 5, 6.

Content

SOME DEFINITIONS AND FORMULAE

- 2D/3D notations:

$$\vec{R} = (\vec{r}, z)$$

$$\vec{\sigma} = (\vec{\mu}, \mu_z)$$

- Fourier transform:

$$FT[f(x)](u) = \tilde{f}(u) = \int f(x) e^{-2i\pi xu} dx$$

- Inverse Fourier transform:

$$FT^{-1}[\tilde{f}(u)](x) = f(x) = \int \tilde{f}(u) e^{+2i\pi xu} du$$

- Dirac function:

$$f(x) = \int f(x')\delta(x - x')dx' \Leftrightarrow (f \star \delta)(x) = f(x)$$

$$\delta(x) = \int e^{+2i\pi xu} du = \int e^{-2i\pi xu} du$$

$$\delta\big(g(x)\big) = \sum_{x_i} \frac{\delta(x - x_i)}{|g'(x_i)|} \text{ with } x_i \text{ simple roots of } g(x). \text{ In particular: } \delta(ax) = \frac{1}{|a|}\delta(x)$$

$$FT[1](x) = FT^{-1}[1](x) = \delta(x)$$

- Complex transmission of a perfect thin lens of focal length f':

$$t_{lens}(\vec{r}) = e^{-i\pi\frac{r^2}{\lambda}\cdot\frac{1}{f'}}$$

- Complex wave front propagating from the plane $z = d$ to the back focal plane $z = f'$ of a lens placed at $z = 0$:

$$U(\vec{r}, f') = \frac{1}{i\lambda f'} e^{\frac{2i\pi(f'-d)}{\lambda}} e^{i\pi\frac{r^2 d}{\lambda f'}\left(\frac{1}{d}+\frac{1}{f'}\right)} \widetilde{U}\left(\frac{\vec{r}}{\lambda f'}, d\right)$$

- In 2D:

$$FT\left[FT[f]\left(\frac{\vec{r}}{a}\right)\right]\left(\frac{\vec{r}}{b}\right) = |a|^2 f\left(-\frac{a}{b}\vec{r}\right)$$

- Discrete Fourier transform of a Λ-periodic signal t and its inverse Fourier transform:

$$\hat{t}_l = \frac{1}{\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-i\frac{2\pi}{\Lambda}lx} dx \leftrightarrow t(x) = \sum_{l\in\mathbb{Z}} \hat{t}_l \, e^{\frac{2i\pi}{\Lambda}lx}$$

Exercises on Chapter 1

Exercise 1.1: the Fresnel approximation

From Eqn (1.4), derive the 2D Fourier transform of $U(\vec{r},z)$.

The idea is to express the 2D Fourier transform of $U(\vec{R})$ in any plane z in the 3D space $\vec{R} = (\vec{r},z)$ with $\vec{r} = (x,y)$ a 2D vector. As in the lecture, one can start from the expression of the 2D Fourier transform of U in the plane $z = 0$:

$$U(\vec{r},z=0) = \int \widetilde{U}(\vec{\mu})e^{2i\pi\vec{\mu}\cdot\vec{r}}\, d^2\vec{\mu} = \int \widetilde{U}(\vec{\mu})e^{2i\pi(\mu_x x+\mu_y y)}\, d\mu_x d\mu_y$$

As $z = 0$, one can introduce the thrid compound of the wave vector $\vec{\sigma} = (\vec{\mu},\mu_z)$.

$$U(\vec{r},z=0) = \int \widetilde{U}(\vec{\mu})e^{2i\pi(\mu_x x+\mu_y y+\mu_z z)}\, d\mu_x d\mu_y = \int \widetilde{U}(\vec{\mu})e^{2i\pi\vec{\sigma}.\vec{R}}\, d^2\vec{\mu}$$

In order for this formula to be valid at $z > 0$, it must satisfy the Helmholtz equation. From the linearity of both the Helmholtz equation and the integral this means that the integrand at all $\vec{\mu}$ must satisfy $|\vec{\sigma}| = \frac{1}{\lambda} = \sigma$ (see lecture notes). This implies:

$$\mu^2 + \mu_z^2 = \sigma^2 \Rightarrow \mu_z = \pm\sqrt{\sigma^2 - \mu^2}$$

Hence the final expression at all z, with the $+$ sign:

$$U(\vec{r},z) = \int \widetilde{U}(\vec{\mu})e^{2i\pi z\sqrt{\sigma^2-\mu^2}}e^{2i\pi\vec{\mu}\cdot\vec{r}}\, d^2\vec{\mu}$$

Which gives the 2D Fourier transform of U in any 2D plane z.

To be noted:

- We chose $+\sqrt{\sigma^2-\mu^2}$ and not $-\sqrt{\sigma^2-\mu^2}$ because we assume in this course that the waves are propagating towards the increasing z (see question 3 for the explanation)
- All these equation are derived for a monochromatic illumination of unvarying wavelength λ
- By definition of the integral and $\vec{\sigma}$, one can see that $U(\vec{r},z)$ is expressed as a continuous summation of the elementary plane waves of complex amplitude $\widetilde{U}(\vec{\mu})$.
- It is implied that $\widetilde{U}(\vec{\mu})$ is the two Fourier transform of U in the plane $z = 0$. To be more rigorous, one could write $\widetilde{U}(\vec{\mu}) = \widetilde{U}(\vec{\mu},z=0)$. Hence, the 2D Fourier transform of $U(\vec{r},z)$ is, according to the expression above:

$$\widetilde{U}(\vec{\mu},z) = \widetilde{U}(\vec{\mu},0)e^{2i\pi z\sqrt{\sigma^2-\mu^2}}$$

Characterize the plane waves for which $|\vec{\mu}| > \sigma$.

$$|\vec{\mu}| > \sigma \Rightarrow \sigma^2 - \mu^2 < 0$$

This implies a purely complex solution for the square root $\sqrt{\sigma^2-\mu^2} = \pm i\sqrt{\mu^2-\sigma^2}$ with $\sqrt{\mu^2-\sigma^2} \in \mathbb{R}^{+*}$. How does the nature choose the solution?

- For $-i\sqrt{\mu^2-\sigma^2}$:

$$\left|\widetilde{U}(\vec{\mu},z)\right| = \left|\widetilde{U}(\vec{\mu},0)e^{+2\pi z\sqrt{\mu^2-\sigma^2}}\right| \underset{z\to+\infty}{\rightarrow} +\infty$$

The Fourier coefficient of frequency $\vec{\mu}$ diverges for increasing z, which clearly raises an energy issue as the energy cannot diverge (Parseval theorem).

- For $+i\sqrt{\mu^2-\sigma^2}$:

$$\left|\tilde{U}(\vec{\mu},z)\right| = \left|\tilde{U}(\vec{\mu},0)e^{-2\pi z\sqrt{\mu^2-\sigma^2}}\right| \underset{z\to+\infty}{\to} 0$$

The Fourier coefficient of frequency $\vec{\mu}$ does not diverge for increasing z. On the contrary it exponentially decreases to zero. These are the so-called evanescent waves that to do not propagate in free space. This delimits the near-field regime. Very specific instrumentations are sensitive to these evanescent waves which are out of the scope of this lecture.

To be remembered: the waves with a wave number satisfying $|\mu| > \sigma$ are evanescent waves and do not propagate in free space.

Second order expansion of a spherical wave.

We admit that the scalar optics expression of a spherical wave is the restriction to the domain of validity of the scalar formalism of the following expression:

$$A_{sph}(\vec{R}) = a_0 \frac{e^{\pm ik_0|\vec{R}-\vec{R_0}|}}{|\vec{R}-\vec{R_0}|}$$

Obviously, the center, $\vec{R_0}$, is not part of the domain of validity of the scalar formalism, but far from the center this expression valid. We shall see later that an aperture limited spherical wave, such as a wave diffracted by a lens with (obviously) a finite aperture does not show a divergence at its center.

What distinguishes a spherical wave diverging from the center from a spherical wave converging to the center?

Without loss of generality, one can center the frame at $\vec{R_0}$. Thus:

$$\vec{R}-\vec{R_0} \to \vec{R}, |\vec{R}-\vec{R_0}| \to |\vec{R}| = R \text{ and } A_{sph}(\vec{R}) = a_0 \frac{e^{\pm ik_0 R}}{R}$$

How to differentiate between diverging and converging waves? One must remember here that for convenient purposes in the lectures, one drops the temporal dependency which is by convention $e^{-i\omega t}$. It only introduces a constant dephasing term. This term disappears for any observable quantity in the context of this lecture (we are only sensitive to the squared modulus of $U \to |U|^2$ averaged along time and $|e^{-i\omega t}| = 1$).

Reintroducing this temporal term in the previous expression, it comes:

$$A_{sph}(\vec{R},t) = a_0 \frac{e^{-i\omega t \pm ik_0 R}}{R}$$

Whose phase term is: $\varphi(\vec{R},t) = -\omega t \pm k_0 R$. A constant phase φ_0 leads to the following equation:

$$R(t) = \frac{\varphi_0 + \omega t}{\pm k_0}$$

Assuming $k_0 > 0$, the distance R increases with t for the $+$ sign and decreases with t for the – sign.

To be remembered: with the temporal convention $e^{-i\omega t}$:

- $+k_0$ indicates diverging spherical wave from $\overrightarrow{R_0}$ and $+\overrightarrow{k_0}.\vec{R}$ a plane wave propagating in the direction of $\overrightarrow{k_0}$.
- $-k_0$ indicates converging spherical wave towards $\overrightarrow{R_0}$ and $-\overrightarrow{k_0}.\vec{R}$ a plane wave propagating in the opposite direction of $\overrightarrow{k_0}$.

Consider a spherical wave centered on axis: $\overrightarrow{R_0} = (0,0,z_0)$. Give its expression in plane $z = 0$

It comes from the equation of a spherical wave, at any position $\vec{R} = (x,y,z)$:

$$A_{sph}\left(\vec{R} = (x,y,z)\right) = a_0 \frac{e^{\pm ik_0|\vec{R}-\overrightarrow{R_0}|}}{|\vec{R}-\overrightarrow{R_0}|} = a_0 \frac{e^{\pm ik_0\sqrt{x^2+y^2+(z-z_0)^2}}}{\sqrt{x^2+y^2+(z-z_0)^2}} = a_0 \frac{e^{\pm ik_0\sqrt{r^2+(z-z_0)^2}}}{\sqrt{r^2+(z-z_0)^2}}$$

In the vicinity of the origin, which is the contact point between the spherical wavefront and its tangent plane on axis, expand the expression of length $|\vec{R}-\overrightarrow{R_0}|$ to the smallest non constant order in $= \frac{|\vec{r}|}{z}$.

In the vicinity of the origin, $r \ll |z - z_0|$ and it comes:

$$|\vec{R}-\overrightarrow{R_0}| = \sqrt{r^2+(z-z_0)^2} = |z-z_0|\sqrt{1+\frac{r^2}{(z-z_0)^2}}$$

$$|\vec{R}-\overrightarrow{R_0}| \simeq |z-z_0|\left(1+\frac{1}{2}\frac{r^2}{(z-z_0)^2}\right) = |z-z_0| + \frac{1}{2}\frac{r^2}{|z-z_0|}$$

Keeping only the smallest relevant terms, which happen to be order zero in the denominator and order 2 in the numerator exponent, express the spherical wave in the following form, which is called the Fresnel approximation:

$A_{sph\_Fresnel}(\vec{r},0) = A_0 e^{-i\frac{\pi r^2}{\lambda|z_0|}}$, and give the expression of constant A_0

Introducing this Taylor expansion in the previous expression, it comes:

$$A_{sph}\left(\vec{R} = (x,y,z)\right) = a_0 \frac{e^{\pm ik_0\left(|z-z_0|+\frac{1}{2}\frac{r^2}{|z-z_0|}\right)}}{|z-z_0|}$$

At $z = 0$ and with $k_0 = \frac{2\pi}{\lambda}$, one gets:

$$A_{sph\_Fresnel}(\vec{r},z=0) = \frac{a_0 e^{\pm i\frac{2\pi}{\lambda}|z_0|}}{|z_0|} e^{\pm i\frac{\pi r^2}{\lambda|z_0|}} = A_0 e^{\pm i\frac{\pi r^2}{\lambda|z_0|}} \text{ with } A_0 = \frac{a_0 e^{\pm i\frac{2\pi}{\lambda}|z_0|}}{|z_0|}$$

To be remembered: As a conclusion, it should be remembered that **any quadratic phase designates a spherical wave**. That is an important conclusion to remember.

To be noted: for a wave propagating towards increasing z,

- it diverges for $z_0 < 0 \Rightarrow +sign \Rightarrow e^{+i\frac{\pi r^2}{\lambda|z_0|}} = e^{-i\frac{\pi r^2}{\lambda z_0}}$
- it converges for $z_0 > 0 \Rightarrow -sign \Rightarrow e^{-i\frac{\pi r^2}{\lambda|z_0|}} = e^{-i\frac{\pi r^2}{\lambda z_0}}$

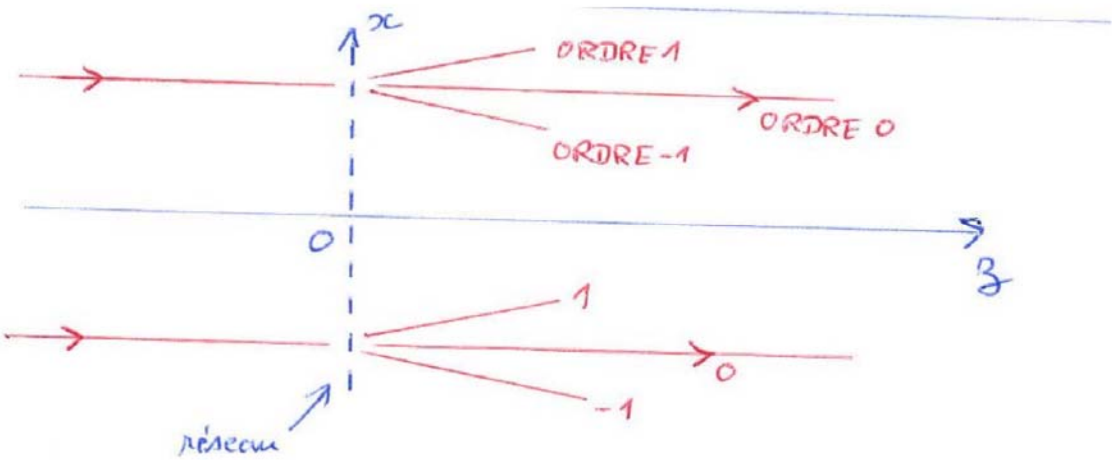
Hence, for the general case of a spherical wave of center $z_0 \in \mathbb{R}$ propagating towards increasing z:

$$A_{sph\_Fresnel}(\vec{r}, z = 0) \propto e^{-i\frac{\pi r^2}{\lambda z_0}}$$

Exercise 1.2 : the Talbot effect

The Talbot effect is a diffraction phenomenon whereby the diffracted orders by a grating spontaneously form an image of the grating after propagating some specific, distance Z_T called the Talbot distance. Here, the Talbot effect will be analytically demonstrated. Typically, it is demonstrated experimentally on the simple case of a "Ronchi ruling", a periodic set of thin slits arranged immediately behind plane Π, of spacing Λ and composed of alternating transparent and opaque bands.

As shown on the figure, the grating is illuminated in normal incidence by a monochromatic plane wave of vacuum wavelength λ



1. Write the Fourier expansion of the grating complex amplitude transmittance (this is just the expression of a Fourier series for a periodic object of period Λ, involving a summation with respect to the "diffraction orders" l).

To be noted: the problem is bi-dimensional: nothing happens along the y-axis. In the following, only the integrals and variables along the x and z-axes will be considered.

The complex transmittance being periodic of period Λ, one can use the expansion in Fourier series rather than the continuous definition:

$$t(x) = \sum_{l\in\mathbb{Z}} \hat{t}_l\, e^{\frac{2i\pi}{\Lambda}lx} = \sum_{l\in\mathbb{Z}} \hat{t}_l\, e^{iKlx} \text{ with } K = \frac{2\pi}{\Lambda}$$

The Fourier transform is given by:

$$\hat{t}_l = \frac{1}{\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x)e^{-iKlx}dx$$

According to the definition of the periodic grating, choosing $x = 0$ at the center of a transparent slit, the complex transmittance for $x \in \left[-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right]$ is given by:

$$t(x) = \begin{cases} 1 \text{ if } x \in \left[-\tau\frac{\Lambda}{2}, \tau\frac{\Lambda}{2}\right] \\ 0 \text{ otherwise} \end{cases} \text{ with } 0 < (1-\tau) < 1 \text{ being the opacity coefficient of the slit.}$$

Hence, $\forall l \in \mathbb{Z}^{\star}$:

$$\hat{t}_l = \frac{1}{\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x)e^{-iKlx}dx = \frac{1}{\Lambda}\int_{-\tau\frac{\Lambda}{2}}^{\tau\frac{\Lambda}{2}} e^{-iKlx}dx = \frac{1}{\Lambda}\left[\frac{e^{-iKlx}}{-iKl}\right]_{-\tau\frac{\Lambda}{2}}^{\tau\frac{\Lambda}{2}} = \frac{e^{iKl\tau\frac{\Lambda}{2}} - e^{-iKl\tau\frac{\Lambda}{2}}}{iKl\Lambda} = 2i\frac{\sin Kl\tau\frac{\Lambda}{2}}{iKl\Lambda}$$

$$\hat{t}_l = \tau\frac{\sin Kl\tau\frac{\Lambda}{2}}{Kl\tau\frac{\Lambda}{2}} = \tau\,\mathrm{sinc}\,\tau l \text{ with } \mathrm{sinc}\,x = \frac{\sin \pi x}{\pi x}$$

Note that as $\mathrm{sinc}\,0 = 1$, this expression is still valid for $l = 0$. As a conclusion:

$$\forall l \in \mathbb{Z}, \hat{t}_l = \tau\,\mathrm{sinc}\,\tau l$$

2. Express the wave vectors of transmitted orders.

For a given l, the wave vector is $\overrightarrow{K_l} = \left(k_{l_x}, k_{l_y}, k_{l_z}\right)$. As mentioned above, nothing happens along the y-axis and $k_{l_y} = 0$. In addition, with the previous notation, $k_{l_x} = Kl$. Moreover, from the previous exercise, $\overrightarrow{k_l}$ must satisfy:

$$\left|\overrightarrow{K_l}\right|^2 = k_{l_x}^2 + k_{l_z}^2 = k_0^2 = \frac{4\pi^2}{\lambda_0^2}$$

Hence: $\overrightarrow{K_l} = \left(Kl, 0, \sqrt{k_0^2 - K^2l^2}\right)$

Note that, as in the previous exercise, the propagating transmitted orders must satisfy:

$$Kl < k_0 \Leftrightarrow l < \frac{\Lambda}{\lambda}$$

3. Expand the z component of those orders to the first non-constant order with respect to spatial frequency (just as was done in the course for the Fresnel approximation, but this time in the spatial frequency domain, as was also done in the course when demonstrating the Huygens-Fresnel "principle"). From there, express the plane wave diffracted in diffraction order l.

For $\lambda \ll \Lambda$, and small values of l, one gets $k_0^2 \gg K^2$ and consequently:

$$\sqrt{k_0^2 - K^2l^2} \simeq k_0\left(1 - \frac{K^2l^2}{2k_0^2}\right) = k_0\left(1 - \frac{\lambda^2l^2}{2\Lambda^2}\right)$$

The diffracted wave in order l is then:

$$U_l(\vec{\mathrm{R}}) = \hat{t}_l e^{i\overrightarrow{K_l}.\vec{\mathrm{R}}} = \tau\,\mathrm{sinc}(\tau l)\, e^{i\left(Klx + k_0\left(1-\frac{\lambda^2l^2}{2\Lambda^2}\right)z\right)} = \tau\,\mathrm{sinc}(\tau l)\, e^{ik_0\left(\frac{l\lambda}{\Lambda}x + \left(1-\frac{\lambda^2l^2}{2\Lambda^2}\right)z\right)}$$

Note that this approximation is not valid for large l. But one can assume that a huge amount of orders satisfy this Taylor expansion to get the main behavior of the diffracted wave behind the slit. The orders that do not satisfy this approximation then produce a second order perturbation of the solution.

4. For which value of propagation distance z is the phase difference between order 0 and order 1 exactly equal to 2π?

The phase φ depends on the order l and the position (x, z):

$$\varphi(x,z,l) = k_0\left(\frac{l\lambda}{\Lambda}x + \left(1 - \frac{\lambda^2 l^2}{2\Lambda^2}\right)z\right)$$

The global complex wave, according to the previous exercise, is:

$$U(\vec{R}) = \sum_{l\in\mathbb{Z}} U_l(\vec{R}) = \sum_{l\in\mathbb{Z}} \tau\,\mathrm{sinc}(\tau l)\, e^{ik_0\left(\frac{l\lambda}{\Lambda}x + \left(1-\frac{\lambda^2 l^2}{2\Lambda^2}\right)z\right)}$$

Note here that for $z = 0$, one naturally finds back:

$$U(x, z = 0) = \sum_{l\in\mathbb{Z}} \tau\,\mathrm{sinc}(\tau l)\, e^{ik_0\frac{l\lambda}{\Lambda}x} = \sum_{l\in\mathbb{Z}} \hat{t}_l e^{\frac{2i\pi}{\Lambda}lx} = t(x)$$

The phase shift introduced by z is $k_0\left(1 - \frac{\lambda^2 l^2}{2\Lambda^2}\right)z$. For a given z, the term $k_0 z$ is constant and has no influence. Let's focus on the other term.

$$k_0\frac{\lambda^2}{2\Lambda^2}z = 2\pi \Leftrightarrow z = \frac{2\Lambda^2}{\lambda}$$

Thus, for $z = \frac{2\Lambda^2}{\lambda}$, $k_0\frac{\lambda^2 l^2}{2\Lambda^2}z = 2l^2\pi$ is multiple of 2π for all $l \in \mathbb{Z}$.

5. Show that in that plane, in the limit of validity of the approximation in question 3, the complex amplitude is a faithful "image" of the original grating.

In that plane, it consequently comes:

$$U\left(x, z = \frac{2\Lambda^2}{\lambda}\right) = \sum_{l\in\mathbb{Z}} \hat{t}_l e^{ik_0\left(\frac{l\lambda}{\Lambda}x+z\right)} e^{i2l^2\pi} = e^{ik_0 z}\sum_{l\in\mathbb{Z}} \hat{t}_l e^{\frac{2i\pi}{\Lambda}lx} = e^{ik_0 z}t(x)$$

As said above, $e^{ik_0 z}$ is just a dephasing term accounting for the propagation on the distance z for a plane of wave number k_0. In optics, it is not measurable by the instruments which are only sensitive to the modulus of U.

Thus, the complex transmittance of the grating is strictly reproduced at the distance $\mathrm{z} = \frac{2\Lambda^2}{\lambda}$, creating a faithful image of it.

To be noted: one can see that this faithful image is shaped periodically along z for entire multiple distances of $\mathrm{z} = n\frac{2\Lambda^2}{\lambda}$.

6. Distance Z_T, equal to half the distance found in question 4, is called the Talbot distance. What is observed in plane $z=Z_T$?

For $z = z_T = \frac{\Lambda^2}{\lambda}$, it comes:

$$U(x, z_T) = \sum_{l\in\mathbb{Z}} \hat{t}_l e^{ik_0\left(\frac{l\lambda}{\Lambda}x+z_T\right)} e^{il^2\pi} = e^{ik_0 z_T}\sum_{l\in\mathbb{Z}} \hat{t}_l e^{2i\pi\left(\frac{lx}{\Lambda}+\frac{l^2}{2}\right)}$$

The above reasoning does not apply here. Indeed for even l, the phase shift is also proportional to 2π, $\Delta\varphi \equiv 0\ [2\pi]$. But for odd l the phase shift is $\Delta\varphi \equiv \pi\ [2\pi]$ and $e^{i\Delta\varphi} = -1$.

Nonetheless, with $x' = x - \frac{\Lambda}{2}$, it comes

$$U(x, z_T) = e^{ik_0 z_T} \sum_{l \in \mathbb{Z}} \hat{t}_l e^{2i\pi\left(\frac{lx'}{\Lambda} + \frac{l}{2} + \frac{l^2}{2}\right)}$$

The phase shift is now $\Delta\varphi = \pi l(l+1)$ which is an even multiple of π for any l: $\Delta\varphi \equiv 0\ [2\pi]$.

Thus:

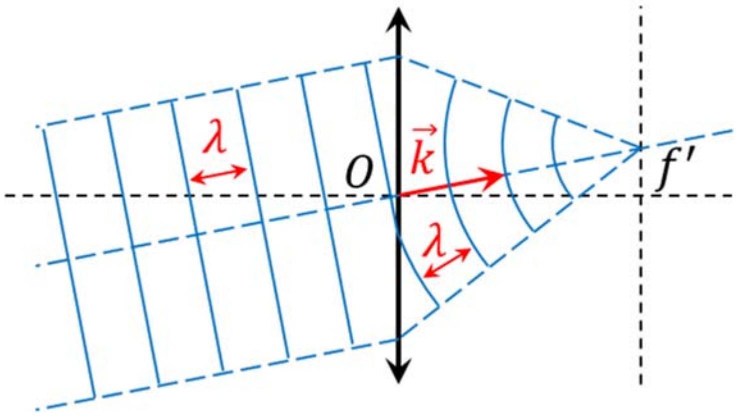
$$U(x, z_T) = e^{ik_0 z_T} \sum_{l \in \mathbb{Z}} \hat{t}_l e^{2i\pi\frac{lx'}{\Lambda}} = e^{ik_0 z_T} t(x') = e^{ik_0 z_T} t\left(x - \frac{\Lambda}{2}\right)$$

As a consequence, on the plane z_T, the diffracted wave shapes an image of the grating shifted of a half period $\frac{\Lambda}{2}$.

Exercise 1.3: the lens as a Fourier transformer

1. Consider a plane wave with wave vector $\vec{k} = k_0\vec{u}$, $\vec{u} = (\sin\alpha \sim \alpha, \sin\beta \sim \beta, \sin\gamma)$ and $|\vec{u}| = 1$.

The following schematic combines the geometrical and physical optics views of a plane wave of wave vector $\vec{k}$ and wavelength λ focused by a lens of focal length f'. The dashed lines account for the geometrical approach in which parallel light rays converge at the same spot on the focal plane in the direction of $\vec{k}$. On the other hand, physical physics focuses on the description of the wave front propagation, emphasized here by the continuous lines. They symbolize phase iso-surfaces, separated by a wavelength.



Where is it focused in the back focal plane of a lens of focal distance (in the image space) f'?

Thus the trigonometric laws give that the position $(\mathrm{x}', \mathrm{y}')$ in the focal plane is:

$$\begin{cases} \tan\alpha = \dfrac{\mathrm{x}'}{f'} \\ \tan\beta = \dfrac{\mathrm{y}'}{f'} \end{cases} \Leftrightarrow \begin{cases} \mathrm{x}' \simeq f'\alpha \\ \mathrm{y}' \simeq f'\beta \end{cases}$$

Using the plane wave expansion of Eqn (1.4), which spatial frequency of the initial complex amplitude $U(\vec{r}, 0)$ is sent to point $\vec{r'} = (x', y', z_F)$ in the focal plane $z = z_F$?

Using the plane wave expansion of a complex incident wave front $U(\vec{R})$

$$U(\vec{r}, z) = \int \tilde{U}(\vec{\mu}) e^{2i\pi z\sqrt{\sigma^2-\mu^2}} e^{2i\pi\vec{\mu}\cdot\vec{r}}\, d^2\vec{\mu}$$

Assuming that the only action of a focusing lens is to make a given plane wave converging at its corresponding focal point, each plane wave $e^{2i\pi z\sqrt{\sigma^2-\mu^2}} e^{2i\pi\vec{\mu}\cdot\vec{r}}$ of amplitude $\tilde{U}(\vec{\mu})$ will focus at $(x'(\vec{\mu}), y'(\vec{\mu}), z_F)$. That is to say the spatial frequency $\vec{\mu} = \frac{1}{\lambda}(\alpha, \beta) = \frac{1}{\lambda f'}(x', y')$ of the initial complex amplitude $U(\vec{r}, 0)$.

To be remembered: a lens shapes an image of the Fourier transform of the incident wave front on its focal plane.

Side question: doesn't the lens itself also contribute to diffraction?

The answer of the previous question was given assuming that the lens itself does not disturb the incident wave front and consequently does not contribute to the diffraction. But in practical, this is not always true.

Indeed, in addition of shaping the beam into spherical waves, a lens introduces a spatial cutoff as its complex transmission drops to $t_{lens}(\vec{r}) = 0$ outside the lens. This modifies the incoming wave front $U(\vec{r},0) \to U(\vec{r},0).t_{lens}(\vec{r})$ and consequently its plane wave extension.

To be remembered: a lens does not contribute to the diffraction if it is big enough not to disturb the incident beam $U(\vec{r},0)$.

In the basic case of a thin object of transmittance $t(\vec{r})$ normally illuminated by a monochromatic plane wave, this means that the Fourier transform of $t(\vec{r})$ is available in the back focal plane. Consider now that the same the object is illuminated by a spherical wave arising from point $(0,0,z_0)$ on axis, where z_o may be positive or negative. Starting back from Eqn 1.5 and expanding the convolution, show which Fourier transform is available in an arbitrary plane z. What happens if z=z_o? Note: from this question, one can see that as far as the observation of the FT goes, at least in the Fresnel approximation, it is equivalent to place a lens of the same focal length just before or just after a diffracting object.

Let's begin by obtaining equation 1.5 in the general case. From:

$$U\left(\vec{R} = (\vec{r},z)\right) = \int \widetilde{U}(\vec{\mu}) e^{2i\pi z\sqrt{\sigma^2-\mu^2}} e^{2i\pi\vec{\mu}\cdot\vec{r}}\, d^2\vec{\mu}$$

A Taylor expansion of $\sqrt{\sigma^2-\mu^2}$ can be performed for $\sigma \gg \mu$.

$$\sqrt{\sigma^2-\mu^2} \simeq \sigma\left(1 - \frac{1}{2}\frac{\mu^2}{\sigma^2}\right)$$

Hence, with $\sigma = 1/\lambda$:

$$U(\vec{R}) \simeq e^{\frac{2i\pi z}{\lambda}} \int \widetilde{U}(\vec{\mu}) e^{-i\pi\lambda z\mu^2} e^{2i\pi\vec{\mu}\cdot\vec{r}}\, d^2\vec{\mu}$$

This is the expression of an inverse 2D Fourier transform:

$$U(\vec{R}) = e^{\frac{2i\pi z}{\lambda}} FT^{-1}\left[\widetilde{U}(\vec{\mu}) e^{-i\pi\lambda z\mu^2}\right](\vec{r})$$

On the other hand, the 1D general formula

$$FT\left[e^{-\pi(\alpha+i\beta)x^2}\right](u) = \frac{1}{\sqrt{\alpha+i\beta}} e^{-\frac{\pi u^2}{(\alpha+i\beta)}}$$

Thus, for $\alpha = 0$ and $\beta = \frac{-1}{\lambda z}$ in 2D, it comes:

$$e^{-i\pi\lambda z\mu^2} = \sqrt{\frac{-i}{\lambda z}}\sqrt{\frac{-i}{\lambda z}} FT\left[e^{i\pi\frac{x^2+y^2}{\lambda z}}\right](\vec{\mu}) = \frac{-i}{\lambda z} FT\left[e^{i\pi\frac{r^2}{\lambda z}}\right](\vec{\mu})$$

In total:

$$U(\vec{R}) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} FT^{-1}\left[FT[U(\vec{r},0)]FT\left[e^{i\pi\frac{r^2}{\lambda z}}\right]\right] = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} FT^{-1}\left[FT\left[U(\vec{r},0) \star e^{i\pi\frac{r^2}{\lambda z}}\right]\right]$$

As introduced in the lecture, $U(\vec{R})$ is given by the following convolution product:

$$U(\vec{R}) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} U(\vec{r},0) \star e^{i\pi\frac{r^2}{\lambda z}}$$

Let's now expand the convolution:

$$U(\vec{R} = (x,y,z)) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} \int U(x',y',0).e^{i\pi\frac{(x-x')^2+(y-y')^2}{\lambda z}} dx'dy'$$

$$U(\vec{R}) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} \int U(\vec{r'},0).e^{i\pi\frac{r^2+r'^2-2xx'-2yy'}{\lambda z}} d^2\vec{r'}$$

$$U(\vec{R}) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} e^{\frac{i\pi r^2}{\lambda z}} \int U(\vec{r'},0).e^{i\pi\frac{r'^2}{\lambda z}} e^{-2i\pi\left(\frac{\vec{r}}{\lambda z}.\vec{r'}\right)} d^2\vec{r'}$$

$$U(\vec{R}) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} e^{\frac{i\pi r^2}{\lambda z}} FT\left[U(\vec{r},0).e^{i\pi\frac{r^2}{\lambda z}}\right]\left(\frac{\vec{r}}{\lambda z}\right)$$

Let's now consider the situation of the present question. For a complex transmittance $t(\vec{r})$ illuminated by a spherical wave front propagating from a point place at $(0,0,z_0)$. According to the results of the exercise 1.1, the expression of $U(\vec{r},0)$ is, dropping the constant complex amplitude of the spherical wave front:

$$U(\vec{r},0) = t(\vec{r}).e^{-i\frac{\pi r^2}{\lambda z_0}}$$

Leading to:

$$U(\vec{r},z) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} e^{\frac{i\pi r^2}{\lambda z}} FT\left[t(\vec{r}).e^{i\pi\frac{r^2}{\lambda}\left(\frac{1}{z}-\frac{1}{z_0}\right)}\right]\left(\frac{\vec{r}}{\lambda z}\right)$$

Thus, on the plane z, the Fourier transform of $t(\vec{r}).e^{i\pi\frac{r^2}{\lambda}\left(\frac{1}{z}-\frac{1}{z_0}\right)}$ is available. In the specific case of $z = z_0$, we find back Fourier transform of $t(\vec{r})$, with a frequency scaling of $\vec{\mu} = \frac{\vec{r}}{\lambda z_0}$. This must be compared to the previous results of a lens with a focal length of f' where $\vec{\mu} = \frac{1}{\lambda f'}\vec{r}$...

Defining a lens as an optical component that transforms an incoming plane wave propagating along its axis into a spherical wave that converges at the focus in the back focal plane, and considering the lens to be a thin object (the expression "thin lens" is common indeed), what is the transmittance of a lens?

Combining the phenomenological results of question 1 and the theoretical results of question 3, it comes that the complex transmittance of lens of focal length f' is:

$$t_{lens}(\vec{r}) = e^{-i\pi\frac{r^2}{\lambda}\cdot\frac{1}{f'}}$$

To be noted: this formula is valid for an infinite lens. Obviously this is not physical. A real lens has a finite aperture outside which $t_{lens}(\vec{r}) = 0$. But it may happen that the incoming wave front is concentrated on an area smaller than the lens pupil and the given formula remains valid.

From that expression, considering the case of a lens illuminated by the spherical wave of question 3 that arises from point $(0,0,z_0)$, demonstrate the (well known) law of thin lenses.

Let's demonstrate the thin lens formula:

$$\frac{1}{d} + \frac{1}{d'} = \frac{1}{f'}$$

For the sake of simplicity, let's place an object of transmittance $t(\vec{r})$ at $z = -d$ and the lens of focal length f' at $z = 0$.

For a plane wave illumination, it comes from the previous results, just before the lens, at $z = 0^-$:

$$U(\vec{r}, 0^-) = \frac{1}{i\lambda d} e^{\frac{2i\pi d}{\lambda}} t(\vec{r}) \star e^{i\pi\frac{r^2}{\lambda d}} = \frac{1}{i\lambda d} e^{\frac{2i\pi d}{\lambda}} e^{\frac{i\pi r^2}{\lambda d}} FT\left[t(\vec{r}).e^{i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{r}}{\lambda d}\right)$$

Just after the "thin" lens, the complex wave front:

$$U(\vec{r}, 0^+) = U(\vec{r}, 0^-).t_{lens}(\vec{r}) = \frac{1}{i\lambda d} e^{\frac{2i\pi d}{\lambda}} e^{\frac{i\pi r^2}{\lambda d}} FT\left[t(\vec{r}).e^{i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{r}}{\lambda d}\right).e^{-i\pi\frac{r^2}{\lambda}\frac{1}{f'}}$$

That is to say:

$$U(\vec{r}, 0^+) = \frac{1}{i\lambda d} e^{\frac{2i\pi d}{\lambda}} e^{\frac{i\pi r^2}{\lambda}\left(\frac{1}{d}-\frac{1}{f'}\right)} FT\left[t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}}\right]\left(\frac{\vec{r}}{\lambda d}\right)$$

And at z:

$$U(\vec{r}, z) = \frac{1}{i\lambda z} e^{\frac{2i\pi z}{\lambda}} e^{\frac{i\pi r^2}{\lambda z}} FT\left[U(\vec{b}, 0^+).e^{i\pi\frac{b^2}{\lambda z}}\right]\left(\frac{\vec{r}}{\lambda z}\right)$$

$$U(\vec{r}, z) = -\frac{1}{\lambda^2 dz} e^{\frac{2i\pi(d+z)}{\lambda}} e^{\frac{i\pi r^2}{\lambda z}} FT\left[e^{\frac{i\pi b^2}{\lambda}\left(\frac{1}{d}-\frac{1}{f'}\right)} FT\left[t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}}\right]\left(\frac{\vec{b}}{\lambda d}\right).e^{i\pi\frac{b^2}{\lambda z}}\right]\left(\frac{\vec{r}}{\lambda z}\right)$$

Dropping the constant and unitary complex terms for a given z:

$$U(\vec{r}, z) \propto FT\left[e^{\frac{i\pi b^2}{\lambda}\left(\frac{1}{d}+\frac{1}{z}-\frac{1}{f'}\right)} FT\left[t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}}\right]\left(\frac{\vec{b}}{\lambda d}\right)\right]\left(\frac{\vec{r}}{\lambda z}\right)$$

For $z = d'$ such as $\frac{1}{d'} + \frac{1}{d} = \frac{1}{f'}$, it comes $e^{\frac{i\pi b^2}{\lambda}\left(\frac{1}{d}+\frac{1}{z}-\frac{1}{f'}\right)} = 1$ and:

$$U(\vec{r}, d') \propto FT\left[FT\left[t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}}\right]\left(\frac{\vec{b}}{\lambda d}\right)\right]\left(\frac{\vec{r}}{\lambda d'}\right)$$

From the definition of the Fourier transform:

$$U(\vec{r}, d') \propto \int FT\left[t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}}\right]\left(\frac{\vec{b}}{\lambda d}\right) e^{-2i\pi\vec{b}.\frac{\vec{r}}{\lambda d'}} d^2\vec{b} = \iint t(\vec{a}).e^{i\pi\frac{a^2}{\lambda d}} e^{-2i\pi\vec{a}.\frac{\vec{b}}{\lambda d}} e^{-2i\pi\vec{b}.\frac{\vec{r}}{\lambda d'}} d^2\vec{a}d^2\vec{b}$$

By inverting the order of the integrals:

$$U(\vec{r},d') \propto \int d^2\vec{a}\left(t(\vec{a}).\,e^{i\pi\frac{a^2}{\lambda d}}\int e^{-2i\pi\vec{a}.\frac{\vec{b}}{\lambda d}}e^{-2i\pi\vec{b}.\frac{\vec{r}}{\lambda d'}}d^2\vec{b}\right)$$

$$U(\vec{r},d') \propto \int d^2\vec{a}\left(t(\vec{a}).\,e^{i\pi\frac{a^2}{\lambda d}}\int e^{-2i\pi\vec{b}.\left(\frac{\vec{a}}{\lambda d}+\frac{\vec{r}}{\lambda d'}\right)}d^2\vec{b}\right) = \int t(\vec{a}).\,e^{i\pi\frac{a^2}{\lambda d}}\delta\left(-\left(\frac{\vec{a}}{\lambda d}+\frac{\vec{r}}{\lambda d'}\right)\right)d^2\vec{a}$$

$$U(\vec{r},d') \propto \int t(\vec{a}).\,e^{i\pi\frac{a^2}{\lambda d}}|\lambda d|\delta\left(\vec{a}+\frac{d}{d'}\vec{r}\right)d^2\vec{a} = |\lambda d|e^{i\pi\frac{dr^2}{\lambda d'^2}}t\left(-\frac{d}{d'}\vec{r}\right)$$

As a conclusion, as expected one gets:

$$U(\vec{r},d') \propto t\left(-\frac{d}{d'}\vec{r}\right)$$

We find back the results of geometrical optics:

- The image of the object is at a distance satisfying: $\frac{1}{d'} = \frac{1}{f'} - \frac{1}{d}$
- The image is reversed: $t(\vec{r}) \rightarrow t(-\vec{r})$
- The image is scaled by the relative distances ratio: $r \rightarrow r' = \left|\frac{d}{d'}\right| r$

With the same idea, it is possible to rapidly obtain another classical formula of geometrical optics with “thin lenses”. Putting two lenses of focal lengths f_1 and f_2 together is equivalent to use a lens with a focal length of f' such as:

$$\frac{1}{f'} = \frac{1}{f_1} + \frac{1}{f_2}$$

Indeed, the incident wave front $U(\vec{r},0^-)$ after the two adjacent lenses becomes:

$$U(\vec{r},0^+) = U(\vec{r},0^-)t_{lens_1}(\vec{r})t_{lens_2}(\vec{r}) = U(\vec{r},0^-)e^{-i\pi\frac{r^2}{\lambda}.\frac{1}{f_1}}e^{-i\pi\frac{r^2}{\lambda}.\frac{1}{f_2}}$$

That is to say:

$$U(\vec{r},0^+) = U(\vec{r},0^-)e^{-i\pi\frac{r^2}{\lambda}.\left(\frac{1}{f_1}+\frac{1}{f_2}\right)} = U(\vec{r},0^-)t_{lens'}(\vec{r})$$

With:

$$t_{lens'}(\vec{r}) = e^{-i\pi\frac{r^2}{\lambda}.\frac{1}{f'}}$$

Exercises on Chapter 3

Exercise 3.1 – Phase in the Optical Fourier transform

The purpose of this exercise is to determine the condition for obtaining the optical Fourier transform in the back focal plane of a thin lens without any spherical phase factor superimposed. The property is valid under the Fresnel approximation only (i.e., spherical waves are quadratic functions in the plane coordinates).

Consider a thin converging lens (L) whose center defines the origin. Its image focal length *f'* and it is illuminated by a monochromatic wave. At plane $z = d \quad (d < 0)$ across the lens axis, the complex amplitude of the illuminating wave is $U(\mathbf{r}, d)$, which may for example have been obtained by a plane wave propagating parallel to the lens axis and illuminating an thin object of complex amplitude transmittance *t(**r**)*.

Reminder: $FT\left(\exp\frac{i\pi\mathbf{r}^2}{a^2}\right) = ia^2 \exp - i\pi\boldsymbol{\mu}^2 a^2$, with $\boldsymbol{\mu}$ the 2D vector (μ_x,μ_y).

1. Express the complex amplitude in plane *z=0₋*, just in front of the lens.

As in the third exercise of the chapter 1, the propagation between the object plane $z = d < 0$ and the lens plane $z = 0$ is given by the Huygens-Fresnel principle:

$$U(\vec{r}, 0^-) = \frac{1}{i\lambda|d|} e^{\frac{2i\pi|d|}{\lambda}} U(\vec{r}, d) \star e^{i\pi\frac{r^2}{\lambda|d|}} = \frac{-1}{i\lambda d} e^{\frac{-2i\pi d}{\lambda}} U(\vec{r}, d) \star e^{-i\pi\frac{r^2}{\lambda d}}$$

To be noted:

- The Huygens-Fresnel formula is based on the distance between the planes of interest. That is why $|d|$ must be used in the formula…
- The convolution written by $U(\vec{r}, d) \star e^{-i\pi\frac{r^2}{\lambda d}}$ is an abusive notation implying that it is estimated in $\vec{r}$. A more rigorous notation would be:

$$U(\vec{r}, d) \star e^{-i\pi\frac{r^2}{\lambda d}} \Leftrightarrow \left(U(\vec{a}, d) \star e^{-i\pi\frac{a^2}{\lambda d}}\right)(\vec{r})$$

2. Express the complex amplitude in plane $z=0_+$, just behind the lens. Here, the lens is assumed to be large enough not to limit the beam diffracted by the object (in optical terms, no "vignetting" occurs).

As seen in chapter 1, the complex transmittance of a perfect lens that does not limit the beam is:

$$t_{lens}(\vec{r}) = e^{-i\pi\frac{r^2}{\lambda}\cdot\frac{1}{f'}}$$

Hence, the complex amplitude in plane $z = 0^+$ is:

$$U(\vec{r}, 0^+) = \frac{-1}{i\lambda d} e^{\frac{-2i\pi d}{\lambda}} \left[U(\vec{r}, d) \star e^{-i\pi\frac{r^2}{\lambda d}}\right] \times e^{-i\pi\frac{r^2}{\lambda}\cdot\frac{1}{f'}}$$

3. Express the complex amplitude in the back focal plane of the lens, *z=f'*.

Similarly than in question 1:

$$U(\vec{\rho},f')=\frac{1}{i\lambda f'}e^{\frac{2i\pi f'}{\lambda}}\left[U(\vec{r},0^+)\star e^{i\pi\frac{r^2}{\lambda f'}}\right](\vec{\rho})$$

And as seen in chapter 1, under the assumption of small angles, it comes the following approximation:

$$\left[U(\vec{r},0^+)\star e^{i\pi\frac{r^2}{\lambda f'}}\right](\vec{\rho})\simeq e^{i\pi\frac{\rho^2}{\lambda f'}}\times TF\left[U(\vec{r},0^+)e^{i\pi\frac{r^2}{\lambda f'}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

Hence, the complex amplitude in plane $z=f'$ is:

$$U(\vec{\rho},f')=\frac{1}{i\lambda f'}e^{\frac{2i\pi f'}{\lambda}}e^{i\pi\frac{\rho^2}{\lambda f'}}\times TF\left[U(\vec{r},0^+)e^{i\pi\frac{r^2}{\lambda f'}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

$$U(\vec{\rho},f')=\frac{-1}{i\lambda d}\frac{1}{i\lambda f'}e^{\frac{-2i\pi d}{\lambda}}e^{\frac{2i\pi f'}{\lambda}}e^{i\pi\frac{\rho^2}{\lambda f'}}\times TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

Let's compute this Fourier transform:

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=\int\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right](\vec{r})e^{-2i\pi\vec{r}.\frac{\vec{\rho}}{\lambda f'}}d^2\vec{r}$$

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=\iint U\left(\vec{r'},d\right)e^{-i\pi\frac{(\vec{r'}-\vec{r})^2}{\lambda d}}e^{-2i\pi\vec{r}.\frac{\vec{\rho}}{\lambda f'}}d^2\vec{r'}d^2\vec{r}$$

And changing the integral order:

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=\int U\left(\vec{r'},d\right)\int e^{-i\pi\frac{r^2+r'^2-2\vec{r}.\vec{r'}}{\lambda d}}e^{-2i\pi\vec{r}.\frac{\vec{\rho}}{\lambda f'}}d^2\vec{r}d^2\vec{r'}$$

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=\int U\left(\vec{r'},d\right)e^{-i\pi\frac{r'^2}{\lambda d}}\int e^{-i\pi\frac{r^2}{\lambda d}}e^{-2i\pi\vec{r}.\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)}d^2\vec{r}d^2\vec{r'}$$

Which makes appear a Fourier transform:

$$\int e^{-i\pi\frac{r^2}{\lambda d}}e^{-2i\pi\vec{r}.\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)}d^2\vec{r}=FT\left[e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)$$

With the reminder given at the beginning of the exercise: $FT\left(\exp\frac{i\pi\mathbf{r}^2}{a^2}\right)=ia^2\exp-i\pi\boldsymbol{\mu}^2a^2$

$$FT\left[e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)d^2\vec{r}=-i\lambda d e^{i\pi\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)^2\lambda d}$$

Hence:

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=-i\lambda d\int U\left(\vec{r'},d\right)e^{-i\pi\frac{r'^2}{\lambda d}}e^{i\pi\left(\frac{\vec{\rho}}{\lambda f'}-\frac{\vec{r'}}{\lambda d}\right)^2\lambda d}d^2\vec{r'}$$

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=-i\lambda d e^{i\pi\frac{d\rho^2}{\lambda f'^2}}\int U\left(\vec{r'},d\right)e^{-2i\pi\frac{\vec{\rho}}{\lambda f'}.\vec{r'}}d^2\vec{r'}$$

$$TF\left[U(\vec{r},d)\star e^{-i\pi\frac{r^2}{\lambda d}}\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)=-i\lambda d e^{i\pi\frac{d\rho^2}{\lambda f'^2}}\widetilde{U}\left(\frac{\vec{\rho}}{\lambda f'},d\right)$$

In total:

$$U(\vec{\rho},f')=\frac{1}{i\lambda f'}e^{\frac{2i\pi(f'-d)}{\lambda}}\,e^{i\pi\frac{\rho^2 d}{\lambda f'}\left(\frac{1}{d}+\frac{1}{f'}\right)}\widetilde{U}\left(\frac{\vec{\rho}}{\lambda f'},d\right)$$

To be noted: for $d=0$, one finds back the formula found in exercise 1.3:

$$U(\vec{R})=\frac{1}{i\lambda f'}e^{\frac{2i\pi f'}{\lambda}}e^{\frac{i\pi r^2}{\lambda f'}}FT\left[U(\vec{r},0).\,e^{i\pi\frac{r^2}{\lambda f'}}\right]\left(\frac{\vec{r}}{\lambda f'}\right)$$

With $U(\vec{r},0)=U(\vec{r},0^-)e^{-i\pi\frac{r^2}{\lambda f'}}$:

$$U(\vec{R}=\vec{\rho},f')=\frac{1}{i\lambda f'}e^{\frac{2i\pi f'}{\lambda}}e^{\frac{i\pi r^2}{\lambda f'}}FT[U(\vec{r},0^-)]\left(\frac{\vec{r}}{\lambda f'}\right)=\frac{1}{i\lambda f'}e^{\frac{2i\pi f'}{\lambda}}e^{\frac{i\pi r^2}{\lambda f'}}\widetilde{U}\left(\frac{\vec{r}}{\lambda f'},0^-\right)$$

4. Under which condition is the Fourier Transform of $U(\mathbf{r},d)$ in that back focal plane devoid of any spherical phase factor? Because of that property, many textbooks on Fourier Optics use it as an illustration of Fourier Optics concepts.

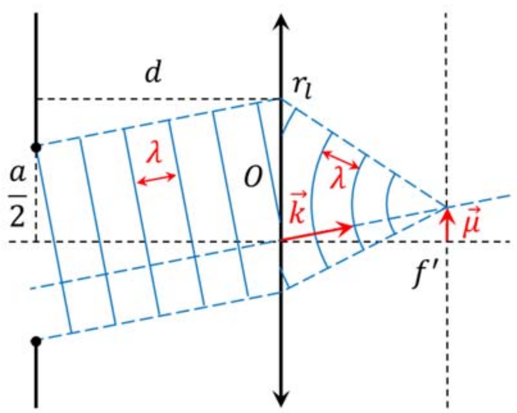
The spherical phase factor is:

$$e^{i\pi\frac{\rho^2 d}{\lambda f'}\left(\frac{1}{d}+\frac{1}{f'}\right)}$$

It vanishes for $d=-f'$, that is to say when the imaged amplitude is placed in the focal plane of the lens.

5. <u>Qualitatively</u>, using a geometrical argument, this questions considers the effect of the lens pupil, i.e. the physical limitation of the lens (usually a mechanical mount holding the lens). Approximately determine the minimal radius for the (circular) lens pupil to let all spatial frequencies with a modulus $|\boldsymbol{\mu}|$ smaller than some given maximal frequency M go unobstructed through the lens if the initial wave $U(\mathbf{r},d)$ is zero outside a given square support $|x|<a/2,|y|<a/2$. Explain why that condition can unfortunately only be an approximation.

Let's do the approximation of the following schematic:



An aperture of radius a decomposes the incoming wave front into a summation of plane wave. As in the third exercise of chapter 1, each plane wave of wave vector $\vec{k} = \frac{2\pi}{\lambda}(\alpha, \beta, \gamma)$ focuses on the focal plane at $\vec{\mu} = \frac{1}{\lambda}(\alpha, \beta)$. The problem being axisymmetric, let's assume $\beta = 0$.

On the schematic, the plane waves are truncated (this is the strong approximation of the geometrical approach which considers that the aperture does not scatter the complex wave front). In order to get all the frequencies with a modulus $|\vec{\mu}| < \mu_{max}$ though the lens, all the truncated plane waves with a wave vector satisfying $|\alpha| < \lambda.\mu_{max}$ must entirely pass through the lens.

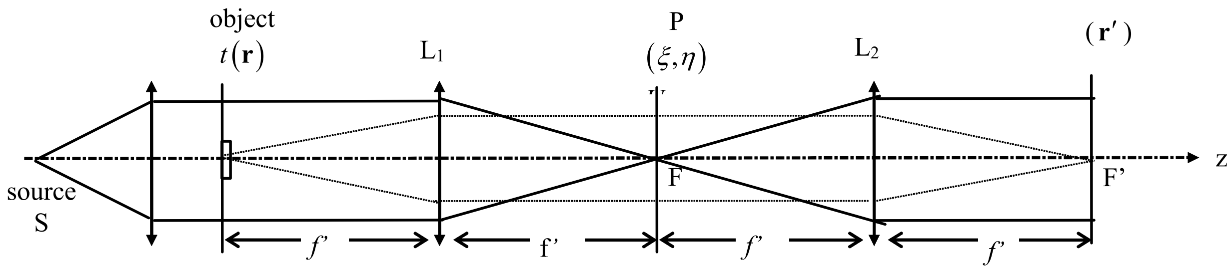
This imposes:

$$r_l > \frac{a}{2} + d|\alpha_{max}| = \frac{a}{2} + d.\lambda.\mu_{max}$$

Exercise 3.2 – Schlieren, phase contrast imaging

(En français, Schlieren se dit "strioscopie").

The following setup is called the "4f" setup. This particular case of coherent spatial frequency filtering is convenient mostly for teaching purposes, as will become clear shortly.



An object of complex amplitude transmittance *t(r)* is illuminated by a plane wave of complex amplitude *Uo*, so that the complex amplitude immediately after the object is $U_o t(\mathbf{r})$. Two lenses L_1 and L_2 of image focal length f' are located as shown on the figure. A pupil (which may just be an empty diaphragm, but in the present case an obstacle will intentionally be inserted) is located in plane P in the back focal plane of L_1, which is also the front focal plane of L_2. Field and aperture limitations in all planes will not be taken into account, i.e. they are considered large enough to avoid any unwanted effect. The running point in the pupil plane *(ξ,η)* is denoted by 2D vector ***ρ***. In questions 1 and 2, the pupil is empty.

3.2.1 – Analysis of the 4f setup :

a. Using the relation between the amplitude just behind the object and the amplitude Up(ρ) in the focal plane of the lens (see exercice 3.1), and the relation between the latter and the amplitude UI(r') in the image plane, show that this calculation accounts for the setup magnification.

In exercise 3.1, one found that the wave front propagating from the plane $z = d$ to the back focal plane $z = f'$ of a lens placed at $z = 0$ is given by:

$$U(\vec{r}, f') = \frac{1}{i\lambda f'} e^{\frac{2i\pi(f'-d)}{\lambda}} e^{i\pi\frac{r^2 d}{\lambda f'}\left(\frac{1}{d}+\frac{1}{f'}\right)} \widetilde{U}\left(\frac{\vec{r}}{\lambda f'}, d\right)$$

Here, with a "4f" setup, $d = -f'$ and consequently, with $k = \frac{2\pi}{\lambda}$:

$$U_P(\vec{\rho}) = \frac{U_0}{i\lambda f'} e^{ik2f'} \tilde{t}\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

$$U_I(\vec{r}) = \frac{1}{i\lambda f'} e^{ik2f'} \widetilde{U}_P\left(\frac{\vec{r}}{\lambda f'}\right)$$

As a consequence:

$$U_I(\vec{r}) = \frac{U_0}{i\lambda f'} e^{ik2f'} FT\left[\frac{1}{i\lambda f'} e^{ik2f'} \tilde{t}\left(\frac{\vec{r}}{\lambda f'}\right)\right]\left(\frac{\vec{r}}{\lambda f'}\right)$$

$$U_I(\vec{r}) = \frac{-U_0}{\lambda^2 f'^2} e^{ik4f'} FT\left[FT[t]\left(\frac{\vec{r}}{\lambda f'}\right)\right]\left(\frac{\vec{r}}{\lambda f'}\right)$$

And in 1D:

$$FT\left[FT[f]\left(\frac{x}{a}\right)\right]\left(\frac{x}{b}\right) = \iint f(x'') e^{-2i\pi x''\frac{x'}{a}} e^{-2i\pi x'\frac{x}{b}} dx'' dx'$$

$$FT\left[FT[f]\left(\frac{x}{a}\right)\right]\left(\frac{x}{b}\right) = \int f(x'') \int e^{-2i\pi x'\left(\frac{x''}{a}+\frac{x}{b}\right)} dx'dx''$$

$$FT\left[FT[f]\left(\frac{x}{a}\right)\right]\left(\frac{x}{a}\right) = \int f(x'')\delta\left(\frac{x''}{a}+\frac{x}{b}\right)dx'' = |a|\int f(x'')\delta\left(x''+\frac{a}{b}x\right)dx'' = |a| f\left(-\frac{a}{b}x\right)$$

And in 2D:

$$FT\left[FT[f]\left(\frac{\vec{r}}{a}\right)\right]\left(\frac{\vec{r}}{b}\right) = |a|^2 f\left(-\frac{a}{b}\vec{r}\right)$$

Thus:

$$U_I(\vec{r}) = -U_0 e^{ik4f'} t(-\vec{r})$$

This accounts for the magnification −1 of this setup.

b. Express $U_p(\rho)$ and $U_I(r')$ using t(r) or its Fourier transform, whichever is simpler.

As a conclusion:

$$U_P(\vec{\rho}) = \frac{U_0}{i\lambda f'} e^{ik2f'} \tilde{t}\left(\frac{\vec{\rho}}{\lambda f'}\right) \text{ and } U_I(\vec{r}) = -U_0 e^{ik4f'} t(-\vec{r})$$

3.2.2 – Direct image without a filter in the pupil plane:

The object is in fact perfectly transparent, but it shows phase variations. For the sake of specificity, it will be assumed to consist of plane parallel plate with a small disk of diameter *D* and thickness *T* acting as a defect that produces a phase delay φ compared to the wave transmitted through the plate substrate.

a. Express ***t(r)*** and show that, when phase delay φ is small, one can write it as the sum of the undisturbed plate transmission s_1 and a wave s_2 whose complex amplitude is proportional to φ and out of phase by π/2 with respect to s_1.

Using the disk function:

$$Disk(\vec{r}) = \begin{cases} 1 \text{ if } r < 1 \\ 0 \text{ otherwise} \end{cases}$$

The complex transmittance is given by:

$$t(\vec{r}) = e^{i\varphi_0} e^{i\varphi Disk\left(\frac{2\vec{r}}{D}\right)}$$

Where φ is the small phase delay introduced by the disk compared to the overall phase delay φ_0 introduced by the glass plate. As the term $e^{i\varphi_0}$ introduces a constant phase in the problem, it can be omitted in the following.

$$t(\vec{r}) = e^{i\varphi Disk\left(\frac{2\vec{r}}{D}\right)} \simeq 1 + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)$$

which is the sum proposed in the question with:

$$s_1 = 1 \text{ and } s_2 = i\varphi Disk\left(\frac{2\vec{r}}{D}\right)$$

Noting that $i = e^{\frac{i\pi}{2}}$, s_2 is indeed out of phase by $\frac{\pi}{2}$ with respect to s_1.

b. Express $U_p(\boldsymbol{\rho})$ and $U_I(\boldsymbol{r'})$ using the disk function $Disk(\mathbf{r}) = \begin{matrix} 1 \text{ if } \mathbf{r}<1 \\ 0 \text{ otherwise} \end{matrix}$ and its Fourier Transform (it will not be necessary to express it explicitly at this stage, although a rough indication of its spatial extent is useful).

From the previous questions, it comes, with $FT\left[Disk\left(\frac{2\vec{r}}{D}\right)\right](\vec{r}) = \tilde{D}(\vec{r})$:

$$U_P(\vec{\rho}) = \frac{U_0}{i\lambda f'} e^{ik2f'} FT\left[1 + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)\right]\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

$$U_P(\vec{\rho}) = \frac{U_0}{i\lambda f'} e^{ik2f'} \left[\delta\left(\frac{\vec{\rho}}{\lambda f'}\right) + i\varphi\tilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]$$

$$U_P(\vec{\rho}) = -iU_0 e^{ik2f'} \left[\delta(\vec{\rho}) + \frac{1}{\lambda f'} i\varphi\tilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]$$

$$U_I(\vec{r}) = -U_0 e^{ik4f'}\left(1 + i\varphi Disk\left(-\frac{2\vec{r}}{D}\right)\right) = -U_0 e^{ik4f'}\left(1 + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)\right)$$

To be noted: $Disk(\vec{r}) = Disk(-\vec{r})$.

c. What is the intensity observed in the image plane?

Normally,

$$U_I(\vec{r}) = -U_0 e^{ik4f'} t(-\vec{r}) \Rightarrow |U_I(\vec{r})|^2 = \left|U_0 e^{i\varphi_0} e^{i\varphi Disk\left(\frac{2\vec{r}}{D}\right)}\right|^2 = |U_0|^2$$

Nonetheless, using the approximated expression of $U_I(\vec{r})$:

$$U_I(\vec{r}) = -U_0 e^{ik4f'}\left(1 + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)\right)$$

One gets $|U_I(\vec{r})|^2 = |U_0|^2 \left|1 + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)\right| > |U_0|^2$. This is an error introduced by the 1st order Taylor expansion…

3.2.3 – Schlieren with a small dark spot

In plane P, on axis, a dark spot is inserted at the convergence point of term s_1. Calculate the complex amplitude $U_{p+}(\boldsymbol{\rho})$ just after the spot and the complex amplitude and intensity in the image. How much is the contrast $C = \frac{I_{defect} - I_{background}}{I_{defect}}$?

Doing so, the term $\delta(\vec{\rho})$ corresponding to s_1 is masked… Then, it comes:

$$U_{P^+}(\vec{\rho}) = \frac{U_0}{\lambda f'} e^{ik2f'} \varphi\tilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)$$

So:

$$U_I(\vec{r}) = \frac{1}{i\lambda f'} e^{ik2f'} \tilde{U}_{P^+}\left(\frac{\vec{r}}{\lambda f'}\right) = \frac{U_0}{i\lambda^2 f'^2} e^{ik4f'} \varphi FT\left[\tilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]\left(\frac{\vec{r}}{\lambda f'}\right)$$

As from above:

$$U_I(\vec{r}) = -U_0 e^{ik4f'} i\varphi Disk\left(\frac{2\vec{r}}{D}\right) \Rightarrow |U_I(\vec{r})|^2 = \varphi^2 |U_0|^2 Disk\left(\frac{2\vec{r}}{D}\right)$$

$$I_{BG} = \left|U_I\left(r > \frac{D}{2}\right)\right|^2 = 0 \text{ and } I_{defect} = \left|U_I\left(r < \frac{D}{2}\right)\right|^2 = \varphi^2 |U_0|^2$$

The resulting contrast is maximal:

$$C = \frac{I_{defect} - I_{BG}}{I_{defect}} = 1$$

To be noted:

- The signal is very small $\propto \varphi^2 \ll 1$.
- The term $I_{defect} = \varphi^2 |U_0|^2$ is correct at the second order in φ because the Taylor expansion at higher terms of the transmittance would lead to higher orders in the resulting intensity.

3.2.4 – Phase contrast with a small transparent spot

Instead of a dark spot, a small phase plate is used, which introduces a $\pi/2$ phase shift at point $\boldsymbol{\rho}$=***0***.

1. Again calculate amplitude $U_{p+}(\boldsymbol{\rho})$ and the complex amplitude and intensity in the image. What are the pros and cons of the phase contrast technique compared to the Schlieren technique ?

Doing so, the term $\delta(\vec{\rho})$ corresponding to s_1 is changed to:

$$\delta(\vec{\rho}) \rightarrow e^{i\frac{\pi}{2}} \delta(\vec{\rho}) = i\delta(\vec{\rho})$$

Then, it comes:

$$U_{P^+}(\vec{\rho}) = \frac{U_0}{\lambda f'} e^{ik2f'} \left[\delta\left(\frac{\vec{\rho}}{\lambda f'}\right) + \varphi \widetilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]$$

And:

$$U_I(\vec{r}) = \frac{1}{i\lambda f'} e^{ik2f'} \widetilde{U}_{P^+}\left(\frac{\vec{r}}{\lambda f'}\right) = \frac{U_0}{i\lambda^2 f'^2} e^{ik4f'} FT\left[\delta\left(\frac{\vec{\rho}}{\lambda f'}\right) + \varphi \widetilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]\left(\frac{\vec{r}}{\lambda f'}\right)$$

As from above:

$$U_I(\vec{r}) = -iU_0 e^{ik4f'} \left[1 + \varphi Disk\left(\frac{2\vec{r}}{D}\right)\right] \Rightarrow |U_I(\vec{r})|^2 = |U_0|^2 \left[1 + \varphi Disk\left(\frac{2\vec{r}}{D}\right)\right]^2$$

Hence:

$$|U_I(\vec{r})|^2 \simeq |U_0|^2 \left(1 + 2\varphi Disk\left(\frac{2\vec{r}}{D}\right)\right)$$

As a conclusion, at the first order:

$$C = \frac{2\varphi}{1 + 2\varphi} \simeq 2\varphi$$

The contrast is at the first order in φ and directly gives the object (the value and the sign of φ).

2. What is changed if the phase plate is partly absorbing: the modulus of its transmission is α, α<*1*? To calculate the contrast, assume that $\alpha \gg \varphi$. Conclude.

Doing so, the term $\delta(\vec{\rho})$ corresponding to s_1 is changed to:

$$\delta(\vec{\rho}) \rightarrow \alpha e^{i\frac{\pi}{2}} \delta(\vec{\rho}) = \alpha i \delta(\vec{\rho})$$

Then, it comes as from above:

$$U_I(\vec{r}) = -iU_0 e^{ik4f'} \left[\alpha + \varphi Disk\left(\frac{2\vec{r}}{D}\right)\right] \Rightarrow |U_I(\vec{r})|^2 = |U_0|^2 \left[\alpha + \varphi Disk\left(\frac{2\vec{r}}{D}\right)\right]^2$$

Hence:

$$|U_I(\vec{r})|^2 \simeq \alpha^2 |U_0|^2 \left(1 + 2\frac{\varphi}{\alpha} Disk\left(\frac{2\vec{r}}{D}\right)\right)$$

As a conclusion, at the first order:

$$C = \frac{2\frac{\varphi}{\alpha}}{1 + 2\frac{\varphi}{\alpha}} \simeq 2\frac{\varphi}{\alpha}$$

Reminding that $\alpha < 1$, the contrast is increased!

3.2.5 – Role of the spot dimension (qualitative discussion)

Consider the Schlieren technique again. This time, the substrate plate size is taken into account: it is a disk of diameter *D'*, and the size of the spot is taken into account as well: it is a disk of diameter *ε*.

a. Express *t(r)*.

The complex transmittance of the substrate plate is given by:

$$t(\vec{r}) = e^{i\varphi_0} Disk\left(\frac{2\vec{r}}{D'}\right) e^{i\varphi Disk\left(\frac{2\vec{r}}{D}\right)}$$

As before, dropping the term $e^{i\varphi_0}$ and using the fact that $D' > D$:

$$t(\vec{r}) \simeq Disk\left(\frac{2\vec{r}}{D'}\right) + i\varphi Disk\left(\frac{2\vec{r}}{D}\right)$$

b. Express U_{P^-}*(**ρ**)* just before the spot. What is the range of appropriate values for *ε*?

As a consequence:

$$U_{P^-}(\vec{\rho}) = -i\frac{U_0}{\lambda f'} e^{ik2f'} \left[\widetilde{D'}\left(\frac{\vec{\rho}}{\lambda f'}\right) + \varphi e^{\frac{i\pi}{2}} \widetilde{D}\left(\frac{\vec{\rho}}{\lambda f'}\right)\right]$$

What was suppressed by playing on the δ at the null frequency is now spread on $\widetilde{D'}\left(\frac{\vec{\rho}}{\lambda f'}\right)$. The Fourier transform of a disk of diameter D is an Airy function, depending on the first order Bessel function. This function cancels for $\mu \simeq 1.22\frac{2}{D}$.

Then the mask in the pupil plane of size ϵ must stop the Airy spot due to the finite size of the object plate while letting untouched the largest part of the Airy spot due to the object. It must consequently obey:

$$1.22\frac{2}{D'} < \epsilon \ll 1.22\frac{2}{D}$$

Exercise 3.3 – Geometrical defocusing

This exercise illustrates the concept of spatial frequency filtering in an optical system where, unlike in the rest of this course, diffraction does not occur (or more precisely, is not taken into account because it always exists but in many cases it is not so conspicuous).

A CMOS camera images a "Siemens star" (en français, mire radiale) composed of N = 36 alternatively black and white sectors, i.e. 18 periods in the angular dimension. The diameter of the Siemens star is *2S* = 20 cm.

The CMOS camera parameters of interest are its image focal length *f'* and the diameter of its circular pupil *2a*. By definition of the "f-number" $f_\#$, $2a=f/f_\#$. The lens is assumed to be thin. For the numerical application, $f_\# = 1.4$

The object is located at a distance *p* from the lens. *p* is a negative quantity, |*p*| is large enough that the image is located nearly, but not exactly, in the back focal plane.

However, there is a slight defocus. The distance from the lens plane to the image is p', but the sensor is located at distance $p'_1 = p'+\varepsilon$ from the lens, which is the image plane for an object that would be located at distance p_1.

On the sensor, the contrast vanishes exactly at the outer boundary of the Siemens star object, and this is the outermost place where the contrast vanishes (by definition not accessible on the sensor, it also vanishes at various places inside the Siemens star defocused image).

Diffraction is neglected, so that the blurred image of a point is a uniform disk. It is reminded that the Fourier transform of the Disk function $Disk(\mathbf{r})$ (= 1 inside the unit radius disk, 0 outside) is $\pi \frac{2J_1(2\pi|\boldsymbol{\rho}|)}{2\pi|\boldsymbol{\rho}|}$, $\boldsymbol{\rho}$ being the Fourier conjugate variable to $\boldsymbol{r}$, and that the first zero of the J1 Bessel function of the first kind, first order is close to 3,83 = 1,22 π.

If p=-5m and f'=3cm, what are the two possible values for p_1?

The next three zeros of J_1 are located at 2,23 π, 3,24 π and 4,25 π. At what other places within the image does the contrast vanish?

The situation is as presented on this figure.

Let us first express the conjugation relations (all distances are algebraic):

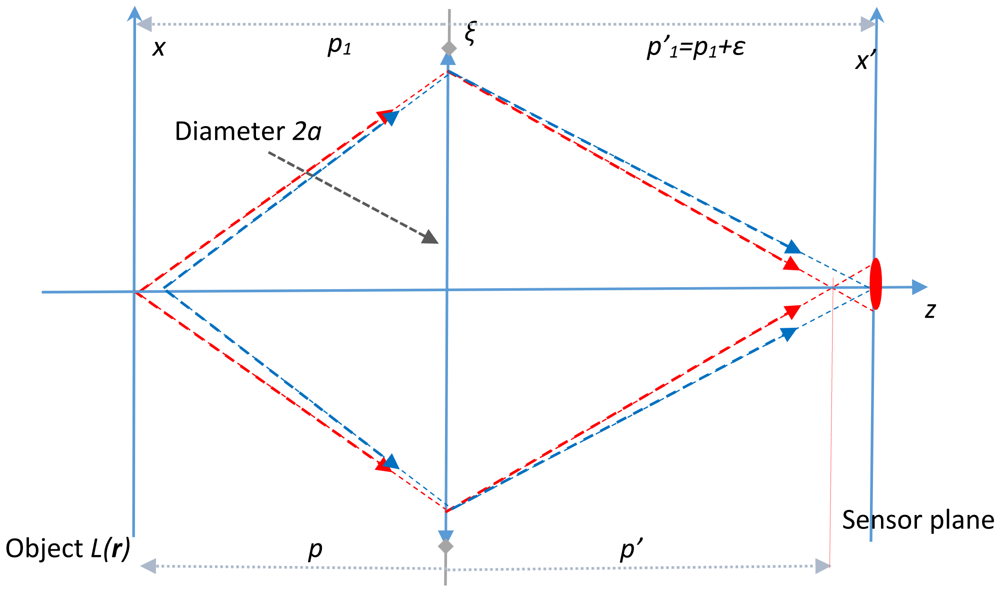
$$\frac{1}{p'} - \frac{1}{p} = \frac{1}{f'}$$

$$\frac{1}{p'_1} - \frac{1}{p_1} = \frac{1}{f'}, \text{ i.e } \frac{1}{p'+\varepsilon} - \frac{1}{p_1} = \frac{1}{f'}$$

which yields

$\frac{1}{p_1} = \frac{1}{p} + \left(\frac{1}{p'+\varepsilon} - \frac{1}{p'} \right)$ and $p_1 = \frac{p}{1 + \frac{p}{p'}\left(\frac{1}{1+\frac{\varepsilon}{p'}} - 1 \right)} \simeq \frac{p}{1 - \frac{p\varepsilon}{p'^2}}$, which further simplifies to

$p_1 \simeq p + \left(\frac{p}{p'} \right)^2 \varepsilon$ in the case where $p\varepsilon \ll p'^2$ (which is not necessarily the case).



The geometrical image is formed at distance p′ from the lens but the sensor is located at distance p′+ε

Let us now consider the red beams in the image space. Since the lens diameter is 2a, the diameter of the image spot is $2r_{spot} = 2a\frac{|\varepsilon|}{p'} = \frac{|\varepsilon|}{f_\#}$ (using the approximation that *p′* is close to *f′*). According to geometrical optics, it is a uniform disk, and the impulse response, which is the image of a single bright point on axis, is, using the notations in the lecture notes:

$$P(\mathbf{r}') = Disk\left(\frac{\mathbf{r}'}{r_{spot}} \right),$$

whence the 'gain' (or 'Transfer Function'), denoted here by *G(μ′)*: $G(\boldsymbol{\mu}') = \pi r_{spot}^2 \frac{2J_1\left(2\pi|\boldsymbol{\mu}'|r_{spot}\right)}{2\pi|\boldsymbol{\mu}'|r_{spot}}$.

(Note: a different normalization for *G(μ′)* will be used later in the course, but this is not relevant here).

In other words the perfect (=well focused) geometrical image of the Siemens star object of luminance L(**r**) in the sensor plane would be, up to a photometric factor: $L'_{focussed}(\mathbf{r}') = L\left(\frac{\mathbf{r}'p}{p'_1} \right)$ - note that this would be obtained with a lens of a slightly different focal length.

Because of the defocus, it is instead:

$$L'_{defocussed}\left(\mathbf{r}'\right)=L'_{focussed}\left(\mathbf{r}'\right)*P\left(\mathbf{r}'\right),$$

and in the Fourier domain,

$$\tilde{L}'_{defocussed}\left(\boldsymbol{\mu}'\right)=\tilde{L}'_{focussed}\left(\boldsymbol{\mu}'\right)G\left(\boldsymbol{\mu}'\right).$$

Let us now express the **local** spatial frequency in the object. The concept of a local spatial frequency is an approximation (it can be refined by using more advanced mathematical concepts, like the Wigner-Ville transform). It consists in considering the object as locally periodic in space. Here, with N black and white sectors, i.e. $N/2$ periods, over a circumference of $2\pi h$ at an arbitrary distance h from the Siemens star center, the local spatial frequency is $N/4\pi h$ in the object space, and therefore $\frac{|p|N}{4\pi h\left(p'+\varepsilon\right)}$ in the image space.

Since the first zero (lowest spatial frequency) corresponds to the outer boundary of the Siemens star, this yields

$2\pi\frac{|\varepsilon|}{f_{\#}}\frac{|p|N}{4\pi Sf'}=1.22\pi$, whence $\varepsilon=\pm 2.44\pi\frac{Sf_{\#}f'}{N|p|}$ and further:

$$p_1=\frac{p}{1\pm\frac{3.83Sf_{\#}}{f'N}}=3.3\text{m or }9.9\text{m}.$$

The contrast vanishes again at the following distances from the center (in the object space): $S\rho_m/\rho_1$, where ρ_m is the m^{th} zero of J_1. Numerically, that gives 5.5 cm, 3.8 cm, 2.9 cm for m=2, 3 and 4 respectively.

Exercise 3.4 – Filtering out halftone print dots in incoherent light

Consider a spatial filtering setup in spatially incoherent, monochromatic light as shown on Figure 1. The notations are the same as in the lecture notes. The vacuum wavelength is λ. The object luminance is $L(\mathbf{r})$.

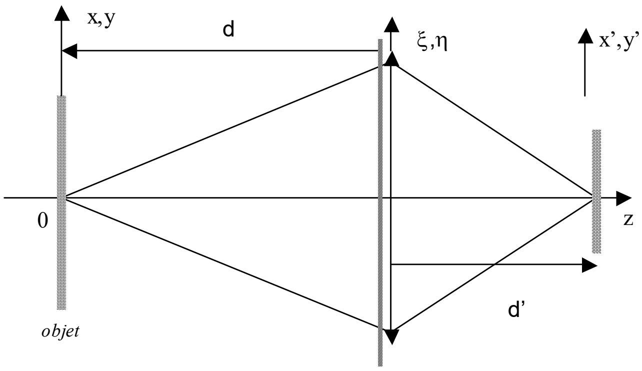


Figure 1.

The lens is thin and has no aberration, its pupil is a disk of radius *a* whose transmittance is denoted by $p_L(\xi,\eta) = p_L(\boldsymbol{\rho})$. Because of some losses, a fraction τ of the energy that has illuminated the pupil reaches the image.

1. Express $p_L(\boldsymbol{\rho})$ using the Disk function defined in the course.

With: $Disk(\vec{r}) = \begin{cases} 1 \text{ if } |\vec{r}| \leq 1 \\ 0 \text{ otherwise} \end{cases}$, it directly comes:

$$p_L(\vec{\rho}) = Disk\left(\frac{\vec{\rho}}{a}\right)$$

2. Express the luminance as transferred to the image, $L'(\mathbf{r}')$, and the image illumination in the geometrical approximation, $\mathcal{E}'_g(\mathbf{r}')$.

With algebraic notations, taking into account the magnification: $\vec{r}' = \frac{d'}{d}\vec{r}$ and, according to the equation (3.16) of the course:

$$L'(\vec{r}') = \tau L(\vec{r})$$

This leads to, using equation (3.16) of the course:

$$\mathcal{E}'_g(\vec{r}') = \frac{\Sigma L'(\vec{r}')}{d'^2} = \frac{\pi a^2}{d'^2} L'(\vec{r}')$$

3. The object is constant along *y* and periodic along *x* with a period $\dfrac{1}{\mu_1}$. Let $\mu'_1 = \mu_1 \dfrac{d}{d'}$,. Express the fundamental frequency and its harmonics in the object and in the image.

The object is $\frac{1}{\mu_1}$-periodic along x and constant along y. Along x, it can be expanded on its Fourier series of fundamental frequency $\vec{\mu}_1 = (\mu_1, 0)$, with harmonics equal to $\vec{\mu}_m = (m\mu_1, 0)$, $m \in \mathbb{Z}$.

The symmetry of the causes leading to a symmetry in the effect, the image will be constant along y and will present a period of $\frac{d'}{d}\frac{1}{\mu_1}$ because of the magnification of the system. Along x, it can be expanded on its Fourier series of fundamental frequency $\vec{\mu}'_1 = (\mu'_1, 0) = \left(\mu_1 \frac{d}{d'}, 0\right)$, with harmonics equal to $\vec{\mu}'_m = (\mu'_m, 0) = (m\mu'_1, 0), m \in \mathbb{Z}$.

4. Writing the object in the form of a Fourier series $L(\mathbf{r}) = \sum_{m\in\mathbb{Z}} L_m \exp 2i\pi\mu_m x$, with the mth harmonic of the object already calculated in question c. Up to question h) below, it will not be necessary to calculate the corresponding Fourier coefficient L_m. Express the image illumination in the presence of diffraction $\mathcal{E}'_d(\mathbf{r}')$ using the incoherent transfer function of the stigmatic instrument with a circular pupil, which, as developed in the course (see the graph in the lecture notes), writes $G_i(\boldsymbol{\mu}) = \frac{2}{\pi}\left(\operatorname{Arccos}\frac{\lambda d'\mu'}{2a} - \frac{\lambda d'\mu'}{2a}\sqrt{1-\left(\frac{\lambda d'\mu'}{2a}\right)^2}\right)$. However, the full expression will not be used here, so the function will just be denoted $G_i(\boldsymbol{\mu})$ or $G_i(\mu_x, \mu_y)$ with μ the modulus of vector $\boldsymbol{\mu}$, whose components are μ_x, μ_y.

As seen in the lectures, $\widetilde{\mathcal{E}'_d}(\vec{\mu}') = \widetilde{\mathcal{E}'_g}(\vec{\mu}') \times G_i(\vec{\mu}')$. Thus:

$$\mathcal{E}'_d(\vec{r}') = \frac{\pi a^2}{d'^2} L'(\vec{r}') = \sum_{m\in\mathbb{Z}} L'_m \, e^{2i\pi\mu'_m x'}$$

With $\mu'_m = m\mu'_1$:

$$L'_m = \begin{cases} \tau L_m \times G_i(m\mu'_1, 0) \text{ if } m\mu'_1 < \dfrac{2a}{\lambda d'} \\ 0 \text{ otherwise} \end{cases}$$

In total:

$$\mathcal{E}'_d(\vec{r}') = \frac{\pi a^2}{d'^2}\tau \sum_{|m|<\frac{2a}{\lambda d'\mu'_1}} L_m \times G_i(m\mu'_1, 0)\, e^{2i\pi m\mu'_1 x'}$$

5. Above which cutoff frequency μ_c (in the <u>object</u> space) is the image $\mathcal{E}'_d(\mathbf{r}')$ uniform?

Looking at the previous sum, the image is uniform if only the constant term of null frequency ($m = 0$) goes though the optical system. That is to say if:

$$\frac{2a}{\lambda d'\mu'_1} < 1 \Leftrightarrow \mu'_1 > \frac{2a}{\lambda d'} \Leftrightarrow \mu_1 > \frac{2a}{\lambda d} = \mu_c$$

6. What is the value of $\mathcal{E}'_d(\mathbf{r}')$ in that case?

In this situation, it directly comes, as $G_i(0,0) = 1$:

$$\mathcal{E}_d(\vec{r}') = \frac{\pi a^2}{d'^2}\tau L_0$$

7. Application: from a train line, a traveler looks from a distance d=10m at a halftone printed black and white picture on an advertisement. Its eyes' pupils are circular with a diameter 2a=5mm. A halftone picture is a set of square dots on a periodic grid with a variable size changing. It will be assumed that the grid is on a square lattice with a period $p = \frac{1}{\mu_1}$ both along x and y. The squares are white on a

black background, their side $c = \alpha p$, $0 \le \alpha \le 1$ changes from dot to dot. The observer, obviously, perceives the whole visible spectrum from the red to the violet. Find the maximal possible value for *p* such that the observer sees the dots as a uniform background.

The visible spectrum spread from the violet $\lambda_v = 400\,nm$ to the red $\lambda_r = 700\text{~}800\,nm$. In the 2D periodic case, the 2D harmonics naturally extend to:

$$\vec{\mu}_{m,n} = \mu_1(m,n) = \frac{1}{p}(m,n)$$

The criterion found in e) to cut all non-null frequencies becomes:

$$\frac{1}{p} > \frac{2a}{\lambda_v d} \Leftrightarrow p < \frac{\lambda_v d}{2a} = 0{,}8\,mm$$

8. In this question, the value of one (and only one) of the coefficients L_m will be needed. In fact, parameter α changes slowly from dot to dot to reproduce variations in the grey level of the initial image. Specify what should be understood by “slowly” in this context. The black background is assumed to be perfectly absorbing and the maximal luminance L_{max} is given by the paper albedo (whiteness) and the primary source (sun, clouds, luminaries, …). Find the law $\alpha(L)$ that will give the appropriate value of α to reproduce the scene luminance L for $0 \le L \le L_{\max}$.

The previous reasoning remains valid as a good approximation if the object can be considered locally periodic so that the local behavior of the image can be determined by the previous results. This is the case if α evolves slowly compared to the period p.

With white squares of side $c = \alpha p$, $0 < \alpha < 1$, the picture is black for $\alpha = 0$ (no white squares) and has a luminance equal to the paper whiteness for $\alpha = 1$ (all is white). According to the previous results, the luminance of the image is given by:

$$L'(\vec{r}') = \tau \sum_{|m,n| < \frac{2a}{\lambda d' \mu_0'}} L_m \times G_i(m\mu_1', n\mu_1')\, e^{2i\pi\mu_1'(mx' + ny')}$$

And keeping only the null frequency:

$$L'(\vec{r}') = \tau L_0 = \tau \frac{1}{p^2} \int_{-\frac{p}{2}}^{\frac{p}{2}} \int_{-\frac{p}{2}}^{\frac{p}{2}} L(x,y) dy\, dx = \tau \frac{1}{p^2} \int_{-\frac{p}{2}}^{\frac{p}{2}} \int_{-\frac{p}{2}}^{\frac{p}{2}} L_{max}.\mathbf{1}_{\left[-\frac{c}{2},\frac{c}{2}\right] \times \left[-\frac{c}{2},\frac{c}{2}\right]}(x,y) dy\, dx$$

$$L'(\vec{r}') = \tau \frac{c^2}{p^2} L_{max} = \tau \alpha^2 L_{max}$$

Then, with $L = L_0$ from the viewing point of the traveler after the spatial filtering:

$$\alpha(L) = \sqrt{L/L_{max}}$$

Exercise 3.5 – Apodising pupils and super-resolving pupils

As has been said in the course, there is not strict mathematical limitation to how close to each other two points should be for an optical instrument to be able to tell them apart. There is only a strict mathematical limitation to how large the period of a periodic object should be to be resolved. In this exercise, we consider the case of two neighboring points and we consider to actions that can be taken to modify the pupil and "improve" their image in some way. In the first case, apodisation, the effect of the secondary maxima in the Airy pattern, which can sometimes hide a weak object in the vicinity of a bright object, is reduced. In the second case, "super-resolving" pupils, the pupil is modified to improve the discrimination of two neighboring points of roughly the same intensity.

The whole exercise is treated in the 1D case. Extension to 2D is straightforward although computationally quite a bit more cumbersome.

a) Apodisation

« Apodisation » (cut the feet, in Greek!) is based on the fact that continuous functions have smoother Fourier transform than discontinuous functions. Compare the uniform pupil

$$p_o(\xi) = \text{rect}\left(\frac{\xi}{\text{a}}\right)$$

with the following cosine-profile apodising pupil:

$$p_1(\xi) = \frac{1}{2}\text{rect}\left(\frac{\xi}{a}\right)\left(1+\cos 2\pi\frac{\xi}{a}\right)$$

- Show that the latter is more continous than the former.

To be noted: all the figures given in this correction are generated with the matlab files TD_5_1 and TD_5_B.

To be noted: the function $\text{rect}(x)$ is defined as:

$$\text{rect}(x) = \begin{cases} 1 \text{ if } |x| < 0.5 \\ 0 \text{ otherwise} \end{cases}$$

As shown on figure A1, the pupil p_0 is discontinuous on its edge. Conversely, p_1 is continuous as well as its first derivative that is equal to 0 on the pupil's edge.

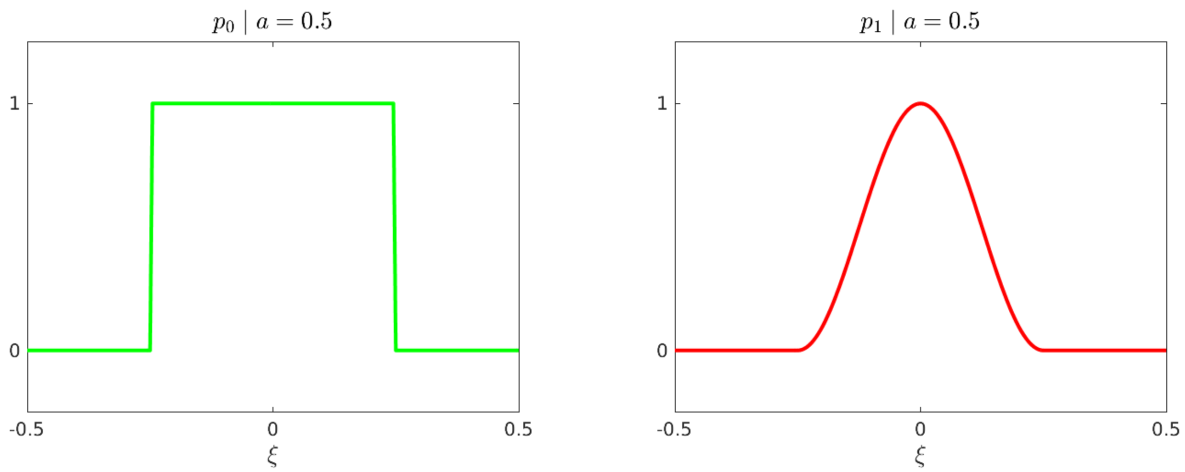


Figure A1. Pupil comparison.

- Write their coherent and incoherent point spread functions.

From the lectures, it comes that the coherent point spread function is proportional to the Fourier transform of the pupil. That is to say in 1D:

$$P_c(x') = \frac{1}{\lambda d'}\tilde{p}\left(\frac{x'}{\lambda d'}\right)$$

Thus, noting:

$$\begin{cases} \alpha_0 = 1 \;\; ; \;\; \alpha_1 = \dfrac{1}{2} \;\; ; \;\; \beta_0 = 0 \;\; ; \;\; \beta_1 = 1 \;\; ; \;\; \gamma_l = 1 \\ p_l(\xi) = \alpha_l . \operatorname{rect}\left(\dfrac{\xi}{a}\right)\left(1 + \beta_l \cos 2\pi\gamma_l \dfrac{\xi}{a}\right) \end{cases}$$

It comes:

$$\widetilde{p_l}(u) = \int p_l(\xi) e^{-2i\pi\xi u} d\xi = \alpha_l \int_{-a/2}^{a/2} \left(1 + \beta_l \cos 2\pi\gamma_l \frac{\xi}{a}\right) e^{-2i\pi\xi u} d\xi$$

$$\widetilde{p_l}(u) = \alpha_l \int_{-a/2}^{a/2} \left(1 + \beta_l \frac{e^{i2\pi\gamma_l \frac{\xi}{a}} + e^{-i2\pi\gamma_l \frac{\xi}{a}}}{2}\right) e^{-2i\pi\xi u} d\xi$$

$$\widetilde{p_l}(u) = \alpha_l \int_{-a/2}^{a/2} \left(e^{-2i\pi\xi u} + \frac{\beta_l}{2}\left(e^{i2\pi\frac{\xi}{a}(\gamma_l - au)} + e^{-i2\pi\gamma_l\frac{\xi}{a}(\gamma_l + au)}\right)\right) d\xi$$

$$\widetilde{p_l}(u) = \alpha_l \left[\frac{e^{-2i\pi\xi u}}{-2i\pi u} + a\frac{\beta_l}{2}\left(\frac{e^{i2\pi\frac{\xi}{a}(\gamma_l - au)}}{i2\pi(\gamma_l - au)} - \frac{e^{-i2\pi\frac{\xi}{a}(\gamma_l + au)}}{i2\pi(\gamma_l + au)}\right)\right]_{-a/2}^{+a/2}$$

$$\widetilde{p_l}(u) = \alpha_l \left[\frac{e^{i\pi au} - e^{-i\pi au}}{2i\pi u} + a\frac{\beta_l}{2}\left(\frac{e^{i\pi(\gamma_l - au)} - e^{-i\pi(\gamma_l - au)}}{2i\pi(\gamma_l - au)} + \frac{e^{i\pi(\gamma_l + au)} - e^{-i\pi(\gamma_l + au)}}{2i\pi(\gamma_l + au)}\right)\right]$$

Noting $\operatorname{sinc} x = \frac{\sin(\pi x)}{\pi x}$:

$$\widetilde{p_l}(u) = a\alpha_l \left[\operatorname{sinc} au + \frac{\beta_l}{2}\left(\operatorname{sinc}(au - \gamma_l) + \operatorname{sinc}(au + \gamma_l)\right)\right]$$

To be noted: this computation can also be done by a convolution:

$$\widetilde{p_l}(u) = \alpha_l . FT\left[\operatorname{rect}\left(\frac{\xi}{a}\right)\right] \star FT\left[1 + \beta_l \cos 2\pi\gamma_l \frac{\xi}{a}\right](u)$$

And it is known that:

$$FT\left[\operatorname{rect}\left(\frac{\xi}{a}\right)\right](u) = a \operatorname{sinc} au$$

$$FT[1](u) = \delta(u)$$

$$FT[\cos 2\pi\nu\xi](u) = FT\left[\frac{e^{2i\pi\nu\xi} + e^{-2i\pi\nu\xi}}{2}\right](u) = \frac{1}{2}\left(\delta(u - \nu) + \delta(u + \nu)\right)$$

Thus:

$$\widetilde{p_l}(u) = \alpha_l.(a \operatorname{sinc} av) \star \left(\delta(v) + \frac{\beta_l}{2}\left(\delta\left(v - \frac{\gamma_l}{a}\right) + \delta\left(v + \frac{\gamma_l}{a}\right)\right)\right)(u)$$

That directly gives the previous result:

$$\widetilde{p_l}(u) = a\alpha_l\left[\operatorname{sinc} au + \frac{\beta_l}{2}(\operatorname{sinc}(au - \gamma_l) + \operatorname{sinc}(au + \gamma_l))\right]$$

Thus, the coherent point spread functions are:

$$P_{c_0}(x') = \frac{a}{\lambda d'}\operatorname{sinc}\frac{ax'}{\lambda d'} \quad \text{and} \quad P_{c_1}(x') = \frac{a}{2\lambda d'}\left[\operatorname{sinc}\frac{ax'}{\lambda d'} + \frac{1}{2}\operatorname{sinc}\left(\frac{ax'}{\lambda d'} - 1\right) + \frac{1}{2}\operatorname{sinc}\left(\frac{ax'}{\lambda d'} + 1\right)\right]$$

The incoherent point spread functions are proportional to the squared modulus of the coherent point spread functions:

$$P_{i_l}(x') \propto \left|P_{c_l}(x')\right|^2$$

Figure A.2 shows the incoherent point spread function of the pupils of figure A1. They are normalized to their highest value. It appears that the full width at half maximum of the second pupil is larger than the one of the first pupil. This may lead to a loss in resolution. Nonetheless, the contrast with the background is increased as the oscillations are extinguished on a smaller distance. The normalization of the curves hides the fact the second pupil is less light than the first one, as shown by the dashed curve. In terms of energy, its transmission is 0.375 lower than the one of the first pupil.

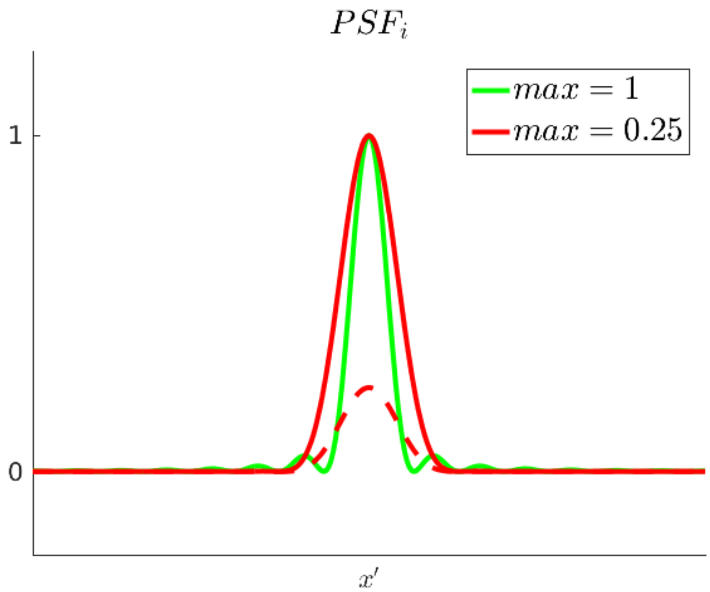


Figure A2. Incoherent point spread functions comparison for the pupils presented in figure A1. The curves are normalized to their maximal value. The dashed curve represents the second pupil point spread function corrected by the relative ratio.

- An object consists of two mutually incoherent points separated by $2.5\frac{\lambda d}{a}$ of which one is 50 times weaker (in flux) than the other (notations of the course). Plot the cross section (along x) of their image.

Figure A3 compares the cross section of the signal produced by the two pupils on a contrast object $L'(x') = \delta(x') + 0.02\delta\left(x' - 2.5\frac{\lambda d'}{a}\right)$.

As expected, the signal (red) of the second object is masked by the second oscillation of the response of the first object (green). Its presence is consequently not obvious in the total signal (black).

With the apodized pupil, the bump produced by the second object becomes clearly visible next to the central peak, even if its full width at half maximum is larger.

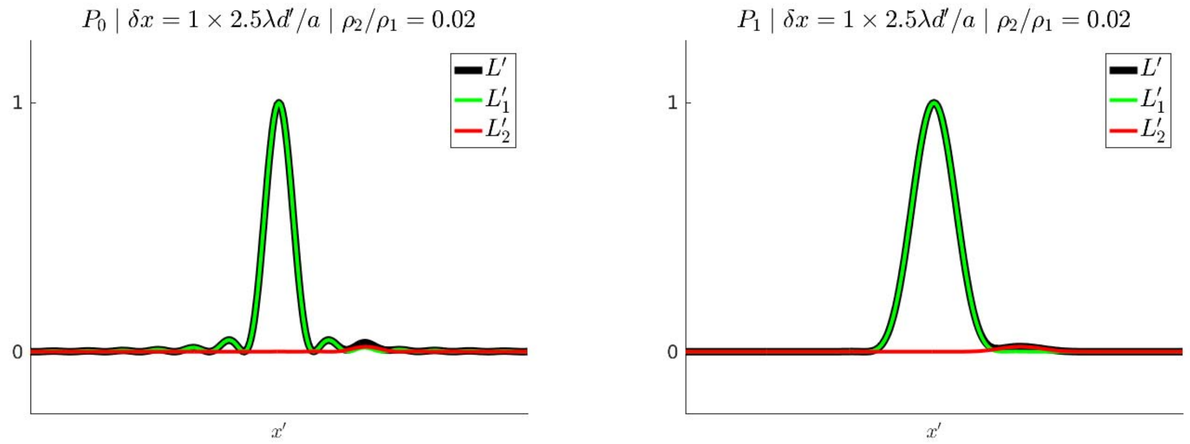


Figure A3. Comparison of the performances of the two pupils of figure A1 applied to a detection of a contrasted object $L'(x') = \delta(x') + 0.02\delta\left(x' - 2.5\frac{\lambda d'}{a}\right)$. The black curves are normalized. The green curve corresponds to the signal produced by the intense object $L'_1(x') = \delta(x')$. The red curve corresponds to the signal produced by the faint object $L'_2(x') = 0.02\delta\left(x' - 2.5\frac{\lambda d'}{a}\right)$

b) "Super-resolution"

As opposed to apodising pupils, “super-resolving” pupils show stronger discontinuities than a uniform pupil. This cannot, obviously, increase the cutoff frequency, but it can sharpen the central peak of the incoherent point spread function at the expense of higher secondary maxima.

Answer the same questions as in a) for the following two pupils:

$$p_o(\xi) = \mathrm{rect}\left(\frac{\xi}{\mathrm{a}}\right)$$

$$p_2(\xi) = \frac{1}{2}\mathrm{rect}\left(\frac{\xi}{a}\right)\left(1 - \cos\pi\frac{\xi}{a}\right)$$

This time, the two mutually incoherent points have the same intensity but are separated by $0.8\frac{\lambda d}{a}$, one being located at the center.

As shown on figure B1, the two pupils p_0 are discontinuous on their edge. But the transmission of p_2 drops to 0 at the center of the pupil plane.

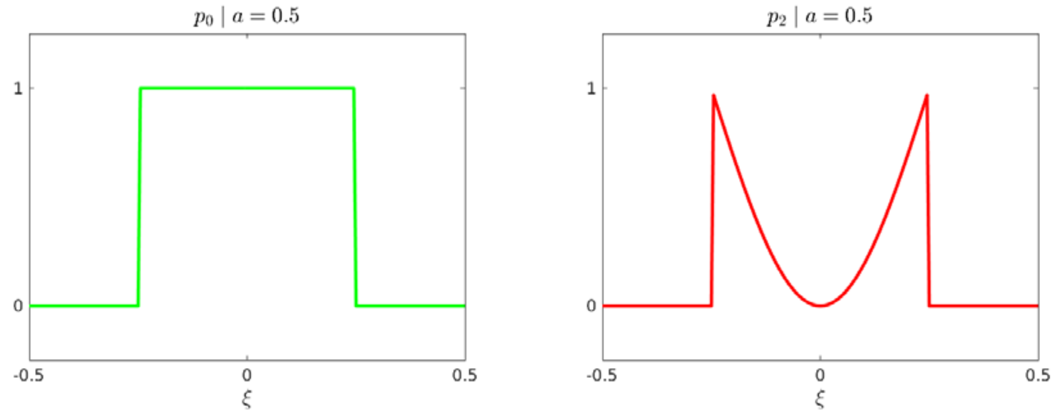


Figure B1. Pupil comparison.

From the previous computation, with $\alpha_2 = \frac{1}{2}$, $\beta_2 = -1$ and $\gamma_2 = \frac{1}{2}$, it comes:

$$\widetilde{p_l}(u) = a\alpha_l \left[\operatorname{sinc} au + \frac{\beta_l}{2}(\operatorname{sinc}(au - \gamma_l) + \operatorname{sinc}(au + \gamma_l))\right]$$

$$P_{c_2}(x') = \frac{a}{2\lambda d'}\left[\operatorname{sinc}\frac{ax'}{\lambda d'} - \frac{1}{2}\operatorname{sinc}\left(\frac{ax'}{\lambda d'} - \frac{1}{2}\right) - \frac{1}{2}\operatorname{sinc}\left(\frac{ax'}{\lambda d'} + \frac{1}{2}\right)\right]$$

The incoherent point spread functions are proportional to the squared modulus of the coherent point spread functions:

$$P_{i_l}(x') \propto \left|P_{c_l}(x')\right|^2$$

Figure B.2 shows the incoherent point spread function of the pupils of figure B1. They are normalized to their highest value. As expected, the central peak of the second pupil is sharpened. Nonetheless, the contrast with the background is lost because the secondary maxima are higher. As previously, the normalization of the curves hides the fact the second pupil is less light than the first one, as shown by the dashed curve.

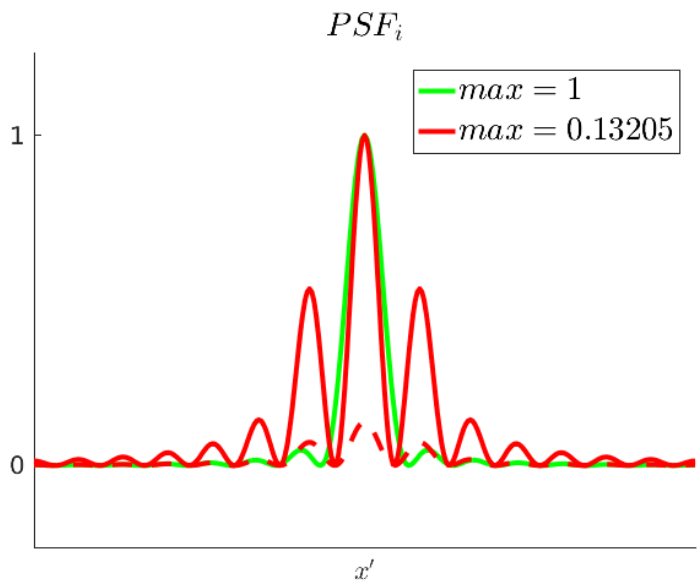


Figure B2. Incoherent point spread functions comparison for the pupils presented in figure B1. The curves are normalized to their maximal value. The dashed curve represents the second pupil point spread function corrected by the relative ratio.

Figure B3 compares the cross section of the signal produced by the two pupils on an object composed of two close sources $L'(x') = \delta(x') + \delta\left(x' + 0.8\frac{\lambda d'}{a}\right)$.

As expected, the signals (red and green) of the two objects are mixed by the first pupil and no characteristic peaks are visible (black). There is no easy criterion to estimate if several sources are present.

With the second pupil, thanks to the sharpener full width at half maximum, two peaks are clearly visible allowing to spate the two sources in the output signal. Nonetheless more secondary peaks are visible which can complicate the interpretation of the cross signal in terms of number of sources.

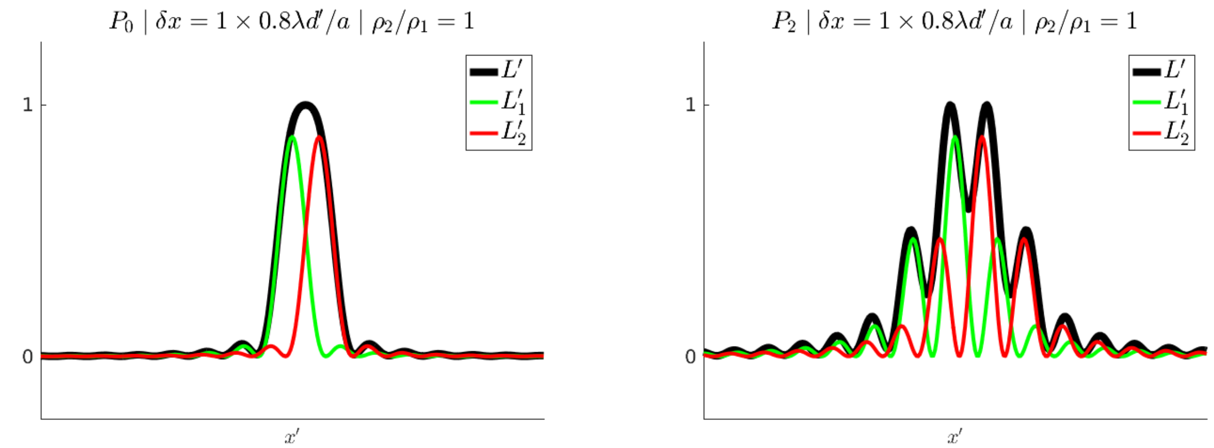


Figure B3. Comparison of the performances of the two pupils of figure B1 applied to a detection of a contrasted object $L'(x') = \delta(x') + \delta\left(x' + 0.8\frac{\lambda d'}{a}\right)$. The black curves are normalized. The green curve corresponds to the signal produced by the intense object $L'_1(x') = \delta(x')$. The red curve corresponds to the signal produced by the faint object $L'_2(x') = \delta\left(x' + 0.8\frac{\lambda d'}{a}\right)$

Exercises on Chapter 5

Exercise 5.1 Diffraction efficiency of a thin grating

The purpose of this exercise is to show that a very simple calculation in the paraxial case leads to a good understanding of diffraction efficiency: the lack of absorption and a suitable phase profile replicating that of the desired wavefront are needed to achieve high efficiencies. Some gratings, indeed, are really well approximated by this simple model.

Consider a 1D-diffraction grating of pitch Λ whose complex amplitude transmittance is $t(x)$. As explained in the course, its diffraction efficiency η_l in order l in the paraxial approximation is the modulus square of the l^{th} coefficient in its Fourier series expansion:
$\eta_l = |t_l|^2 = \left| \frac{1}{\Lambda} \int_{x_o}^{x_o+\Lambda} t(x) \exp \frac{-2i\pi l x}{\Lambda} dx \right|^2$ (Note: x_o is arbitrary).

1. What is the diffraction efficiency of a "Ronchi ruling", a binary component used for the illustration of elementary courses and also used in wavefront testing, consisting of alternating absorbing and transparent bands with a duty cycle f (the transparent bands have a width $f\Lambda$). What is the maximum value that η_l can reach, and when does that happen?

To be noted: the problem is bi-dimensional: nothing happens along the y-axis. In the following, only the integrals and variables along the x and z-axes will be considered.

As in exercise 1.2, the complex transmittance being periodic of period Λ, one can use the expansion in Fourier series rather than the continuous definition:

$$t(x) = \sum_{l \in \mathbb{Z}} t_l \, e^{\frac{2i\pi}{\Lambda} l x} = \sum_{l \in \mathbb{Z}} t_l \, e^{iKlx} \text{ with } K = \frac{2\pi}{\Lambda}$$

The Fourier transform is given by:

$$t_l = \frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-\frac{2i\pi}{\Lambda} l x} dx = \frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-iKlx} dx$$

According to the definition of the periodic grating, choosing $x = 0$ at the center of a transparent slit, the complex transmittance for $x \in \left[-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right]$ is given by:

$$t(x) = \begin{cases} 1 \text{ if } x \in \left[-f\frac{\Lambda}{2}, f\frac{\Lambda}{2}\right] \\ 0 \text{ otherwise} \end{cases} \text{ with } 0 < (1-f) < 1 \text{ being the opacity coefficient of the slit.}$$

Hence, $\forall l \in \mathbb{Z}^*$:

$$t_l = \frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-iKlx} dx = \frac{1}{\Lambda} \int_{-f\frac{\Lambda}{2}}^{f\frac{\Lambda}{2}} e^{-iKlx} dx = \frac{1}{\Lambda} \left[\frac{e^{-iKlx}}{-iKl} \right]_{-f\frac{\Lambda}{2}}^{f\frac{\Lambda}{2}} = \frac{e^{iKlf\frac{\Lambda}{2}} - e^{-iKlf\frac{\Lambda}{2}}}{iKl\Lambda} = 2i \frac{\sin Klf\frac{\Lambda}{2}}{iKl\Lambda}$$

$$t_l = f\frac{\sin Klf\frac{\Lambda}{2}}{Klf\frac{\Lambda}{2}} = f\,\mathrm{sinc}\,fl \text{ with } \mathrm{sinc}\,x = \frac{\sin \pi x}{\pi x}$$

Note that as $\mathrm{sinc}\,0 = 1$, this expression is still valid for $l = 0$. As a conclusion:

$$\forall l \in \mathbb{Z}, t_l = f\,\mathrm{sinc}\,fl$$

To be noted: There is a limited number of orders that can be transmitted by the grating. Indeed, for a given l, the wave vector is $\overrightarrow{K_l} = \left(k_{l_x}, k_{l_y}, k_{l_z}\right)$. As mentioned above, nothing happens along the y-axis and $k_{l_y} = 0$. In addition, with the previous notation, $k_{l_x} = Kl$. Moreover, for a monochromatic propagation, $\overrightarrow{k_l}$ must satisfy:

$$\left|\overrightarrow{K_l}\right|^2 = k_{l_x}^2 + k_{l_z}^2 = k_0^2 = \frac{4\pi^2}{\lambda_0^2}$$

Hence: $\overrightarrow{K_l} = \left(Kl, 0, \sqrt{k_0^2 - K^2 l^2}\right)$ and the propagating transmitted orders must satisfy:

$$Kl < k_0 \Leftrightarrow l < \frac{\Lambda}{\lambda}$$

Thus, $\eta_l = f^2\,\mathrm{sinc}^2\,fl = \frac{\sin^2 \pi fl}{\pi^2 l^2}$. That is to say, $\eta_0 = f^2$ and $\forall |l| \geq 1, \eta_l \leq \frac{1}{\pi^2} \simeq 10\ \%$. And:

$$\eta_l = \frac{1}{\pi^2} \Leftrightarrow \begin{cases} l = \pm 1 \\ f = \frac{1}{2} \end{cases}$$

2. Same question for a pure phase grating: the bands are all transparent but alternate between phase 0 and phase φ_o. Here, of course, φ_o can be optimized as well. How is φ_o related to the etch thickness?

In this question, the complex transmittance for $x \in \left[-\frac{\Lambda}{2}, \frac{\Lambda}{2}\right]$ is given by:

$$t(x) = \begin{cases} e^{i\varphi_o} \text{ if } x \in \left[-f\frac{\Lambda}{2}, f\frac{\Lambda}{2}\right] \\ 1 \text{ otherwise} \end{cases}$$

Thus, noting t_R the transmission of the Ronchi ruling of the previous question, it comes:

$$t(x) = 1 + \left(e^{i\varphi_o} - 1\right) t_R(x)$$

With $\frac{1}{\Lambda}\int_{-\Lambda}^{\Lambda} 1 e^{-iKlx} dx = \begin{cases} 1 \text{ if } l = 0 \\ 0 \text{ if } |l| \geq 1 \end{cases}$, it comes from the previous calculus:

$$t_l = \begin{cases} 1 + \left(e^{i\varphi_o} - 1\right) f & \text{if } \ l = 0 \\ \left(e^{i\varphi_o} - 1\right) f\,\mathrm{sinc}\,fl & \text{if } \ |l| \geq 1 \end{cases}$$

And the efficiency is:

$$\forall |l| \geq 1, \eta_l = \left|e^{i\varphi_o} - 1\right|^2 \frac{\sin^2 \pi fl}{\pi^2 l^2} = 4\sin^2\frac{\varphi_0}{2}\frac{\sin^2 \pi fl}{\pi^2 l^2}$$

As previously, the maximum is reached for:

$$\eta_{max} = \frac{4}{\pi^2} \Leftrightarrow \begin{cases} l = \pm 1 \\ f = \frac{1}{2} \\ \varphi_0 \equiv \pi \ [2\pi] \end{cases}$$

It can be noted that the efficiency is 4 times higher than for the Ronchi ruling! 40% of the energy goes into the order +1 and another 40% in the order -1.

The relation of etch thickness e and the phase shift φ_0 is linked with the refractive index of the surrounding medium n_0 (generally the air) and the refractive index of the grating n by the following formula:

$$\varphi_0 = \frac{2\pi}{\lambda}(n - n_0)e$$

3. (optional) what is the diffraction efficiency of sinusoidal grating $t(x) = \frac{1}{2}\left(1 + \cos\frac{2\pi x}{\Lambda}\right)$?

Similarly to the exercise 3.5:

$$t_l = \frac{1}{\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} \frac{1}{2}\left(1 + \cos 2\pi\frac{x}{\Lambda}\right) e^{-\frac{2i\pi}{\Lambda}lx} dx = \frac{1}{2\Lambda}\int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} \left(1 + \frac{e^{i2\pi\frac{x}{\Lambda}} + e^{-i2\pi\frac{x}{\Lambda}}}{2}\right) e^{-\frac{2i\pi}{\Lambda}lx} dx$$

$$t_l = \frac{1}{2\Lambda}\left[\frac{e^{-\frac{2i\pi}{\Lambda}lx}}{-\frac{2i\pi}{\Lambda}l} + \frac{1}{2}\left(\frac{e^{i2\pi\frac{(1-l)}{\Lambda}x}}{i2\pi\frac{(1-l)}{\Lambda}} - \frac{e^{-i2\pi\frac{(1+l)}{\Lambda}x}}{i2\pi\frac{(1+l)}{\Lambda}}\right)\right]_{-\Lambda/2}^{+\Lambda/2}$$

$$t_l = \frac{1}{2}\left[\frac{e^{i\pi l} - e^{-i\pi l}}{2i\pi l} + \frac{1}{2}\left(\frac{e^{i\pi(1-l)} - e^{-i\pi(1-l)}}{2i\pi(1-l)} + \frac{e^{i\pi(1+l)} - e^{-i\pi(1+l)}}{2i\pi(1+l)}\right)\right]$$

$$t_l = \frac{1}{2}\left[\frac{\sin\pi l}{\pi l} + \frac{1}{2}\left(\frac{\sin\pi(1-l)}{\pi(1-l)} + \frac{\sin\pi(1+l)}{\pi(1+l)}\right)\right]$$

Thus, $t_{\pm 1} = \frac{1}{2}\left[\frac{\sin\pi}{\pi} + \frac{1}{2}\operatorname{sinc} 0\right] = \frac{1}{4}$. The efficiency is: $\eta_{\pm 1} = \frac{1}{16} = 6.25\ \%$.

To be noted: the result of this question can be obtained in one line… Indeed:

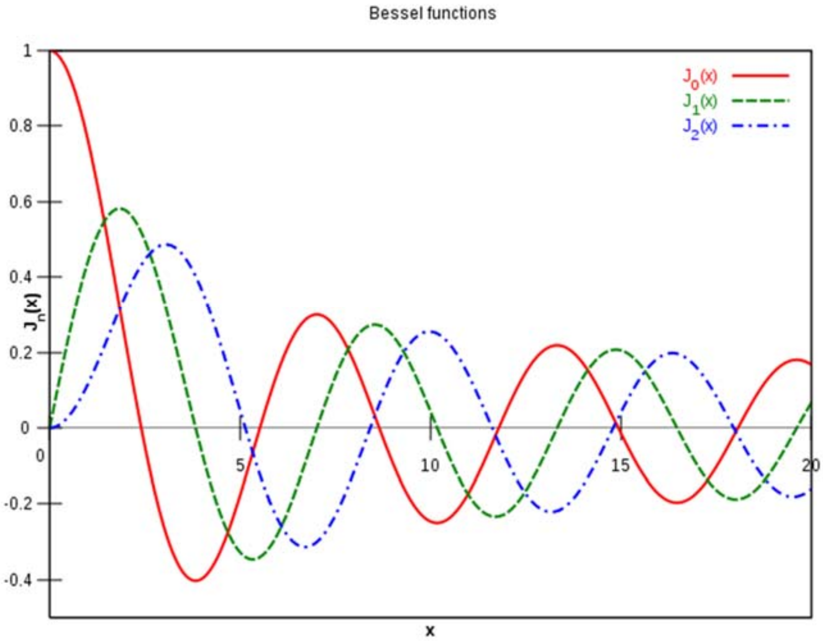
$$t(x) = \frac{1}{2}\left(1 + \cos 2\pi\frac{x}{\Lambda}\right) = \frac{1}{2} + \frac{1}{4}e^{i2\pi\frac{x}{\Lambda}} + \frac{1}{4}e^{-i2\pi\frac{x}{\Lambda}}$$

By uniqueness of the Fourier series $(x) = \sum_{l\in\mathbb{Z}} t_l\, e^{\frac{2i\pi}{\Lambda}lx}$, one directly gets that:

$$t_0 = \frac{1}{2} \ \text{ and } \ t_{\pm 1} = \frac{1}{4} \ \text{ and } \ t_l = 0 \ \text{ otherwise}$$

4. (optional) what is the maximal diffraction efficiency of a grating whose phase is sinusoidal: $t(x) = \exp i\phi(x)$ with $\phi(x) = \phi_o \cos\frac{2\pi x}{\Lambda}$?

Reminder: the Bessel functions of the first kind, represented graphically in the figure, obey the following identity: $\exp(im\cos\varphi) = \sum_{l\in\mathbb{Z}} i^l J_l(m) \exp il\varphi$



The complex transmittance is:

$$t(x) = e^{i\phi(x)} = e^{i\phi_0 \cos\frac{2\pi}{\Lambda}x} = \sum_{l\in\mathbb{Z}} i^l J_{|l|}(\phi_0) e^{il\frac{2\pi}{\Lambda}x} = \sum_{l\in\mathbb{Z}} t_l \, e^{\frac{2i\pi}{\Lambda}lx}$$

By uniqueness of the Fourier series:

$$t_l = i^l J_{|l|}(\phi_0) \Rightarrow \eta_l = J_{|l|}^2(\phi_0)$$

For $l \neq 0$, the Bessel function that culminates the highest is the first order Bessel function at ~ 0.6 for $\phi_0 \simeq 1.9$. The maximal efficiency is consequently $\eta_{max} = \eta_{\pm 1} = 36\ \%$.

5. Show that no grating whose complex amplitude transmittance is an even function can exceed a diffraction efficiency of 50% in any order (this question is trivial).

The symmetry of the causes leading to symmetry in the effects, an even transmittance function leads to:

$$\eta_l = \eta_{-l}$$

In addition, the sum of the efficiency satisfies:

$$\sum_{l\in\mathbb{Z}} \eta_l \leq 1 = 100\ \%$$

It implies that:

$$\forall |l| > 1, \eta_l + \eta_{-l} = 2\eta_l \leq 100\ \% \Rightarrow \eta_l \leq 50\ \%$$

Only the 0-order can have an efficiency above $50\ \%$...

6. Show that no grating whose complex amplitude transmittance is a real function can exceed a diffraction efficiency of 50% in any order (this question is trivial as well).

For a real transmittance, it directly comes:

$$t_l^\star = \left(\frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-\frac{2i\pi}{\Lambda} lx} dx \right)^\star = \frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t^\star(x) e^{+\frac{2i\pi}{\Lambda} lx} dx = \frac{1}{\Lambda} \int_{-\frac{\Lambda}{2}}^{\frac{\Lambda}{2}} t(x) e^{-\frac{2i\pi}{\Lambda}(-l)x} dx = t_{-l}$$

Thus,

$$t_l^\star = t_{-l} \Rightarrow \eta_l = \eta_{-l}$$

As above, only the 0-order can have an efficiency above 50 %...

7. Identify the only thin grating that can reach 100% diffraction efficiency in some order other than zero.

Let l_0 be the order reaching 100 % efficiency. It means that:

$$\begin{cases} \eta_{l_0} = 1 \\ \eta_{l \neq l_0} = 0 \end{cases} \Rightarrow \begin{cases} t_{l_0} = e^{\frac{2i\pi\varphi_0}{\Lambda}}, \varphi_0 \in \mathbb{R} \\ t_{l \neq l_0} = 0 \end{cases}$$

Thus, the only thin grating that can reach 100 % efficiency in some order other than zero as a complex transmittance equal to:

$$t(x) = \sum_{l \in \mathbb{Z}} t_l e^{\frac{2i\pi}{\Lambda} lx} = t_{l_0} e^{\frac{2i\pi}{\Lambda} l_0 x} = e^{\frac{2i\pi}{\Lambda}(l_0 x + \varphi_0)}$$

This is a perfectly phase ramp with an offset of $\frac{2\pi\varphi_0}{\Lambda}$. Let's note here that this is equivalent to the phase introduced by a mirror.

Exercise 5.2 Gratings fabricated by multilevel binary masking

Consider a grating of pitch Λ consisting of rectangular bands etched in glass, anti-reflection coated on both sides.

a) Only one etched band per period:

(this is almost the same as question 1 of exercise 5.1). In this question, the grating consists of a single transparent band from *x*=x_1 to *x*=x_2, etched on a thickness *h*. The rest is opaque. Write the Fourier series expansion of the complex amplitude transmittance.

In this situation, the transmittance is not null only between x_1 and x_2. In that case, a phase shift is introduced, due to the refractive index gap $\delta n = n - n_0$ between the refractive index n of the grating and the refractive index n_0 of the surrounding medium. This phase shift is equal to: $\varphi_h = \frac{2\pi}{\lambda} h\delta n$.

$$t(x) = \mathrm{rect}\left(\frac{x - \frac{1}{2}(x_2 + x_1)}{x_2 - x_1}\right) e^{i\varphi_h} = \mathrm{rect}\left(\frac{x - x_c}{\delta x}\right) e^{i\varphi_h}$$

With: $\delta x = x_2 - x_1$ and $x_c = \frac{x_2+x_1}{2}$.

As in the previous exercise:

$$t(x) = \sum_{l\in\mathbb{Z}} t_l e^{\frac{2i\pi}{\Lambda}lx} \quad \text{with} \quad t_l = \frac{1}{\Lambda}\int_0^{\Lambda} t(x) e^{-\frac{2i\pi}{\Lambda}lx} dx$$

And it comes:

$$t_l = \frac{e^{i\varphi_h}}{\Lambda}\int_{x_1}^{x_2} 1 e^{-\frac{2i\pi}{\Lambda}lx} dx = \frac{e^{i\varphi_h}}{\Lambda}\left[\frac{e^{-\frac{2i\pi}{\Lambda}lx}}{-\frac{2i\pi}{\Lambda}l}\right]_{x_1}^{x_2} = \frac{e^{i\varphi_h}}{\Lambda}\frac{e^{-\frac{2i\pi}{\Lambda}lx_2} - e^{-\frac{2i\pi}{\Lambda}lx_1}}{-\frac{2i\pi}{\Lambda}l}$$

$$t_l = \frac{e^{i\varphi_h} e^{\frac{2i\pi}{\Lambda}lx_c}}{\Lambda}\delta x \frac{e^{-\frac{i\pi}{\Lambda}l\delta x} - e^{+\frac{i\pi}{\Lambda}l\delta x}}{-\frac{2i\pi}{\Lambda}l\delta x}$$

In total:

$$t_l = \frac{\delta x}{\Lambda} e^{2i\pi\left(\frac{lx_c}{\Lambda} + \frac{h\delta n}{\lambda}\right)} \mathrm{sinc}\frac{l\delta x}{\Lambda}$$

b) M step grating:

This time, each period consists of *M* successive bands, *j*=0 to *M*-1, of heights h_j, located between abscissas x_j and x_{j+1}, with x_0=0 and x_M=Λ. Express the diffraction efficiency of all orders.

By linearity of the Fourier series, it comes that the new Fourier coefficient t_l is the sum of all the contribution t_l^j of the elementary parts between x_j and x_{j+1}, where t_l^j is given by the previous question:

$$t_l^j = \frac{x_{j+1} - x_j}{\Lambda} e^{2i\pi\left(\frac{l}{\Lambda}\cdot\frac{x_j + x_{j+1}}{2} + \frac{h_j\delta n}{\lambda}\right)} \mathrm{sinc}\frac{l(x_{j+1} - x_j)}{\Lambda}$$

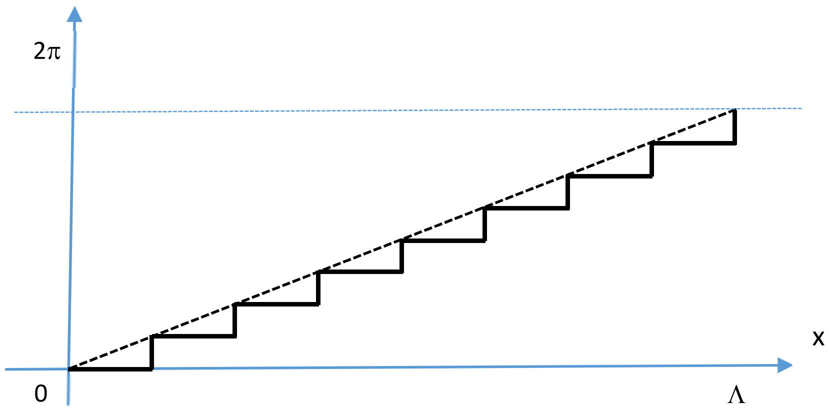
It comes:

$$t_l = \sum_{j=0}^{M-1} \frac{x_{j+1} - x_j}{\Lambda} \operatorname{sinc} \frac{l(x_{j+1} - x_j)}{\Lambda} e^{2i\pi\left(\frac{l}{\Lambda} \cdot \frac{x_j + x_{j+1}}{2} + \frac{h_j \delta n}{\lambda}\right)}$$

And $\eta_l = |t_l|^2$

c) Echelette grating :

The grating of question b is aimed at approximating an échelette grating, i.e. a prism whose effect is a blaze at wavelength λ_o, as shown on the figure. Express the diffraction efficiency. What is the maximal achievable efficiency if the grating consists of $M' = \log_2 M$ etching steps. Applications: M'=1, 2, 3, 4.



In this question, the grating is blazed at λ_0. It means that for the wavelength λ_0, the maximum height at the end of the period introduces a phase shift to 2π. That is to say a thickness of $\frac{\lambda_0}{\delta n}$. Thus:

$$\forall j \in [0, M-1], x_{j+1} - x_j = \frac{\Lambda}{M} \quad \text{and} \quad \frac{x_j + x_{j+1}}{2} = \frac{\Lambda}{M}\left(\frac{1}{2} + j\right) \quad \text{and} \quad h_j = \frac{j}{M} \frac{\lambda_0}{\delta n}$$

Consequently:

$$t_l = \sum_{j=0}^{M-1} \frac{1}{M} \operatorname{sinc} \frac{l}{M} e^{2i\pi\left(\frac{l}{M}\left(\frac{1}{2}+j\right)+\frac{j}{M}\frac{\lambda_0}{\lambda}\right)} = \frac{e^{i\pi\frac{l}{M}}}{M} \operatorname{sinc} \frac{l}{M} \sum_{j=0}^{M-1} e^{\frac{2i\pi}{M} j\left(l+\frac{\lambda_0}{\lambda}\right)}$$

This is a geometrical sum:

$$t_l = \frac{e^{i\pi\frac{l}{M}}}{M} \frac{1 - e^{2i\pi\left(l+\frac{\lambda_0}{\lambda}\right)}}{1 - e^{\frac{2i\pi}{M}\left(l+\frac{\lambda_0}{\lambda}\right)}} \operatorname{sinc} \frac{l}{M}$$

Thus:

$$\eta_l = \frac{1}{M^2} \left| \frac{1 - e^{2i\pi\left(l+\frac{\lambda_0}{\lambda}\right)}}{1 - e^{\frac{2i\pi}{M}\left(l+\frac{\lambda_0}{\lambda}\right)}} \right|^2 \operatorname{sinc}^2 \frac{l}{M} = \frac{1}{M^2} \frac{\sin^2 \pi\left(l + \frac{\lambda_0}{\lambda}\right)}{\sin^2 \frac{\pi}{M}\left(l + \frac{\lambda_0}{\lambda}\right)} \operatorname{sinc}^2 \frac{l}{M}$$

For the order $l \leq 0$, with $\lambda \simeq \frac{\lambda_0}{l}$, it comes that:

$$\sin^2 \pi \left(l + \frac{\lambda_0}{\lambda}\right) \sim \pi^2 \left(l + \frac{\lambda_0}{\lambda}\right) \quad \text{and} \quad \sin^2 \frac{\pi}{M}\left(l + \frac{\lambda_0}{\lambda}\right) \sim \frac{\pi^2}{M^2}\left(l + \frac{\lambda_0}{\lambda}\right)$$

Thus:

$$\eta_l = \operatorname{sinc}^2 \frac{l}{M}$$

For the first order $l = -1$ (and so $\lambda \simeq \lambda_0$), the efficiency is:

$$\eta_{-1} = \operatorname{sinc}^2 \frac{1}{M}$$

This gives the following table:

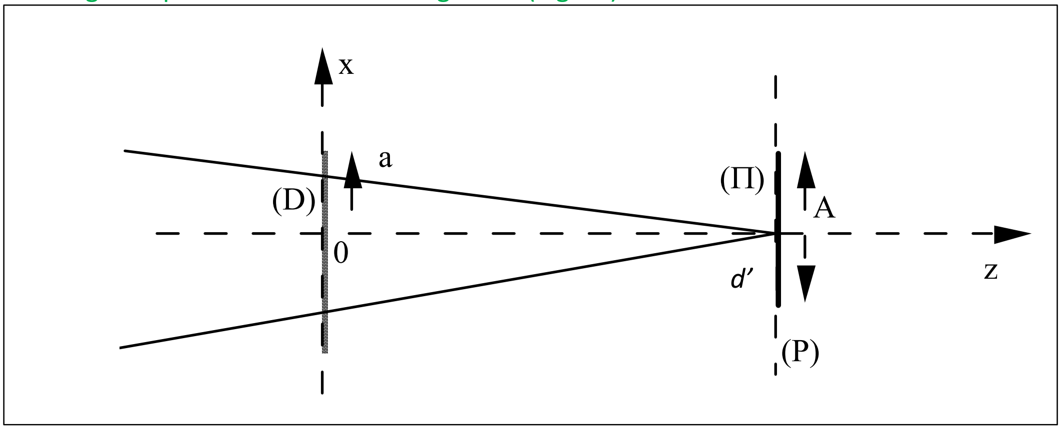
M'	$1 \rightarrow M = 2$	$2 \rightarrow M = 4$	$3 \rightarrow M = 8$	$4 \rightarrow M = 16$
η_{-1}	$\sim 40.5\ \%$	$\sim 81.1\ \%$	$\sim 95.0\ \%$	$\sim 98.7\ \%$

The efficiency increases rapidly with the number of etching steps.

Let's note here that these results must be put into perspective with the previous exercise. Indeed, with $M \rightarrow \infty$, the proposed etched grating tends towards the phase ramp that was deduced in question 7: the only grating with an efficiency of $100\ \%$ in one order.

EXERCISE ON CHAPTER 6: DOUBLE PHOTOGRAPHY OF SPECKLE AND STELLAR SPECKLE INTERFEROMETRY

A strong and thin diffuser (*D*) is described by complex amplitude transmittance $t(x,y)$. It is illuminated by a spherical laser wave of vacuum wavelength λ converging in plane (*P*), a distance *d'* behind (*D*). An image sensor (*Π*) records the speckle in plane (*P*). The illuminated area on the diffuser is bound by a circular pupil of radius *a*. The sensor area is a square of side A centered about the convergence point of the illuminating wave (Figure).



"1) White diffuser":

At the scale of sensor (*Π*), the diffuser can be considered "white", which means that it would take a larger sensor to observe a departure of (*D*) from "whiteness". Using the above defined parameters and in particular the statistical properties of function *t*, explain what that statement means.

That statement means that the diffuser complex amplitude transmittance autocorrelation, properly scaled, is smaller than the Fourier transform of the sensor area, i.e. smaller than $\lambda d'/A$.

2) Description of the speckle:

a) the speckle on the sensor is fully developed: what does that statement mean? (Note: it is equivalent to saying that the diffuser is "strong").

b) Give an order of magnitude of the speckle grain size.

a) the statistical average of the speckle complex amplitude is zero (there is no direct light, no "zero order). b) because the diffuser is limited by a disk-shaped pupil of radius *a*, the speckle grain size is on the order of $\lambda d'/2a$.

3) The Fourier transform of the recorded speckle pattern:

The sensor responds linearly to the incoming illuminance. The pixels are much smaller than the speckle grains, so that each pixel essentially samples the illuminance at its center. Describe qualitatively and sketch on a figure the Fourier transform of the speckle pattern recorded by the sensor: non scattered part, grain size of the scattered part, spatial variation of the scattered part. The choice of units is free, provided they are consistent and well specified.

The speckle pattern is itself a random function, but its illuminance is a real positive function. It is therefore analogous to a weak diffuser, that shows a strong non scattered part. The scattered part creates in the Fourier transform the analog of a speckle pattern whose envelope (in Fourier coordinates, i.e. spatial frequencies) extends over $2a/\lambda d'$, the inverse of the speckle size in the image, and whose grain is on the order of $1/A$.

4) Double exposure:

In a further experiment, instead of just measuring the speckle illuminance on the sensor, two exposures are overlaid on the sensor:

- First exposure: the same as before, during an exposure time *T*.
- The sensor is moved by an amount x_o in direction x in its own plane (*P*).
- Second exposure: the speckle is measured again during the same exposure time *T* as before.

Express the time integrated illuminance at a pixel that was initially located at (*x,y*). Note: the photometric quantity corresponding to the time integration of the illuminance (illuminance x exposure time) is known as "lumination".

Let us denote by $E(x,y)$ the speckle illumination on the sensor during the first exposure. The illumination during the second exposure is therefore $E(x-x_o,y)$. The time integrated lumination is therefore $L(x,y)=T\left[E(x,y)+E(x-x_o,y)\right]$.

5) Fourier transform of the doubly exposed sensor data:

a) Compared to question 3, what is changed in the speckle Fourier transform?

b) Show that that change allows to measure x_o.

c) What is, as an order of magnitude, the minimal value of xo below which the method cannot be expected to provide accurate results?

a) In question 3, we had only the Fourier transform of $E(x,y)$. b) The Fourier transform of $L(x,y)$ is, apart from the unessential factor *T*, $\tilde{E}(\mu_x,\mu_y)(1+\exp 2i\pi\mu_x x_o)$. Its squared modulus is $4\left|\tilde{E}(\mu_x,\mu_y)\right|^2\cos^2\pi\mu_x x_o$, which shows fringes of period (in the μ_x variable) $1/x_o$. c) If the period is larger than the extent of the speckle pattern $2a/\lambda d'$, it is difficult to measure, therefore xo should be larger than $\lambda d'/2a$, one speckle grain size, for the translation to be easily measurable.

6) Application: measuring a double star in the presence of atmospheric turbulence ("stellar speckle interferometry", Labeyrie 1970):

a) In Labeyrie's stellar speckle interferometry method to measure double stars, a star (in fact, possibly a double star) illuminates a telescope of focal length *f'*. Atmospheric turbulence generates speckle. The telescope diameter is D, the atmosphere is a strong diffuser, and its "grain" size, known in astronomy as the "Fried parameter", is r_o. Qualitatively describe the image observed for a short exposure (too short for the atmosphere to change), which in fact is a speckle pattern.

Just like in question 2, the speckle grain size is $\lambda f'/D$. However, this time the diffuser is not "white": its "grain size" is r_o. Therefore, the speckle extends over an area $\lambda f'/r_o$.

b) In many cases, the exposure time is much too long for the atmosphere to be considered motionless: the speckle pattern is completely blurred. What is the angular resolution ε (known as the "seeing").

In that case, the speckle is blurred and only its envelope appears. The image size is therefore $\lambda f'/r_o$ and the angular resolution is $\varepsilon = \lambda/r_o$.

c) In that case, obviously, a double star whose two components are spaced by an angle θ significantly smaller than ε cannot be distinguished. However, it is possible to use the sensor with an exposure time small enough that the atmosphere does not change – the detector is linear, so even small signals can be recorded even though they are corrupted by quantization noise (an effect that will be neglected here). Qualitatively describe the observed image in that case. For simplicity, the two components of the double star will be considered of equal intensity.

The image consists of two speckle patterns of size $\lambda f'/r_o$, with a speckle grain size $\lambda f'/D$, shifted by $\theta f'$

d) Qualitatively describe the Fourier transform of the sensor data and identify the noise limit.

Just like in question 5, the Fourier transform consists of a non-scattered part, of no interest, and a scattered part of extent $D/\lambda f'$ with grains of size $r_o/\lambda f'$, modulated by fringes of period $1/\theta f'$. In the limit of low noise, a star separation $\theta < \lambda/r_o$ can be detected. However, if the fringe period is larger than the speckle extent, the fringes cannot be seen, therefore the method is useful to detect double stars with a separation $\lambda/D < \theta < \lambda/r_o$. Because of the short exposure, the light level is low, the fringes are in fact composed of a limited number of photon impact superimposed on detector noise.

e) To reduce the effect of noise, several short exposures are acquired, separately Fourier transformed, and the Fourier transforms are added. Show that this allows to estimate the angular separation θ of the double star with a better signal to noise ratio than with one single exposure.

For multiple exposures, the noise is not the same, the fringe period is the same, therefore by superimposing <u>not</u> multiple speckle patterns, but multiple Fourier transforms of speckle patterns, the noise is averaged and the fringes appear out of the noise, potentially leading to a resolution limit equal to the diffraction limit of the whole telescope, λ/D .