Chapter 3 – Speckle

Chapter 2 is a reminder on random variables and random processes. It is not part of the course taught during the classes and there will be no examination on it, it is provided for the purpose of optional reading. Part of the lecture on October 25 will be presented by Nicolas Vedrenne on speckle from atmospheric turbulence. It complements the present chapter.

Recommended textbooks:

- J.C. Dainty, ed., "Laser Speckle and Related Phenomena", collection Topics in Applied Physics, volume 9, Springer, Berlin, 1975 (several re-editions).
- J.W. Goodman, "Statistical Optics", John Wiley, New York, 1985, pages 347-464.
- J.W. Goodman, “Speckle Phenomena, Theory and Applications”, Roberts & Co, Publisher, 2006.
- M. Françon, « La granularité laser (speckle) et ses applications en Optique », Masson, Paris, 1978.

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I – INTRODUCTION TO SPECKLE, INVESTIGATION OF A SIMPLE CASE

1.1 – Inhomogeneities create speckle

Speckle appears when light illuminates randomly non homogeneous media. It appears in the form of a randomly arranged set of grains on a dark background. Grains in a speckle pattern are not faithful images of grains that may exist in the random medium, e.g. an image of sand pebbles on a beach reproduces the sand pebbles and should not be called speckle. Well defined grains in an object like sand pebbles are referred to as “graininess” (French granulation). Speckle arises when imaging grains that are not resolved by the imaging instrument, or when light is observed in an arbitrary plane which is not conjugated to the grainy object. In French, usually “speckle” is just called “speckle”. Standard French would argue for “granularité”. However, for decades, astronomers have designated the specific kind of speckle that arises from atmospheric turbulence by the old French name “tavelures”.

As we just mentioned, speckle may be observed in the image of a non-homogeneous medium, when the “grains” of the medium are not resolved, or in a diffraction figure. The non-homogeneous medium is often called a “diffuser”. These two kinds of speckle have in fact rather different properties. Coherence also has a major impact on the phenomenon.

In the case of speckle in the (Fresnel of Fraunhofer) diffraction pattern of an inhomogeneous medium, there is no clear distinction between diffraction and scattering: light is diffracted, or scattered, by the inhomogeneities in the medium. Scattering in many cases refers to random media, while diffraction is preferred for deterministic cases, but exception exist, e.g. “Mie scattering” by a spherical droplet. The most usual cases of speckle include:

- scattering by a rough surface illuminated by a laser,
- image of a rough surface illuminated by a laser, when the inhomogeneities of the surface are not resolved,

- scattering by a rough surface illuminated by a source of small spatial extent, including in some case the sun shining on a piece of paper, in which case speckle appears as a set of low-contrast coloured grains,
- propagation of light through the turbulent atmosphere, that severely degrades the image of objects in the sky,
- modal noise in multimode optical fibres.

1.2 – Statistical description of a diffuser:

Speckle is normally described by its statistical properties. Therefore, it is appropriate to introduce the probability density functions for the amplitude, phase, and irradiance of the speckle field, as well as their auto- and cross-correlations, and possibly their higher order moments (see Chap I).

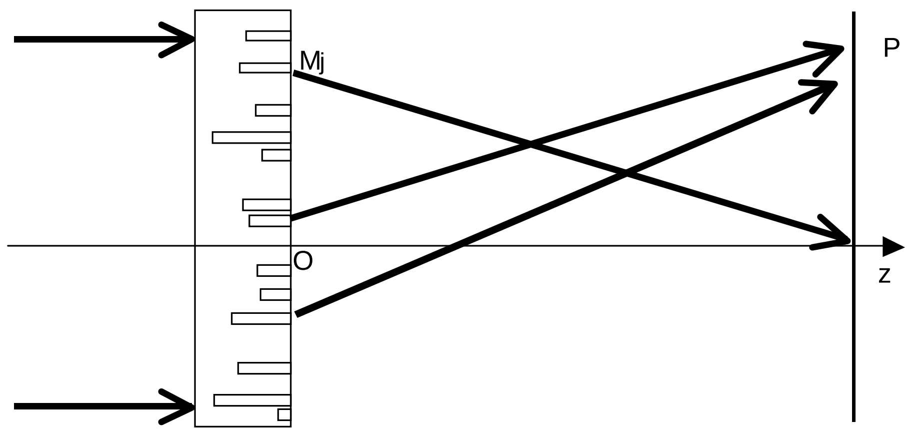
Scattering mostly consists of random phase delays $\phi(\mathbf{r})$ between various points **r** imposed by the scattering medium, although in some cases of lesser interest only amplitude modulations exist. The basic quantity will therefore be that spatial phase. Comparing speckle with coherence, the transition from random processes of time to random processes of space should be mentioned.

The statistical properties of $\phi(\mathbf{r})$ depend on the category of scattering object considered: Some of the distinctions applicable are listed here.

- Weak diffuser or strong diffuser: a diffuser is called "weak" if it introduces phase variations on the order of or smaller than 2π. It is called "strong" if those phase variations are large compared to 2π.
- "White" diffuser and "correlated" diffuser: the distinction here is between grains that are so small that their autocorrelation can be assimilated to a Dirac peak at the scale considered, and grains large enough that the size of their autocorrelation should be taken into account.
- Amplitude only diffuser, phase only diffuser.
- Thick and thin diffuser: see the Fourier Optics course, a thin diffuser is described by an amplitude transmittance $t(\mathbf{r}) = |t(\mathbf{r})| \exp i\phi(\mathbf{r})$ which does not (significantly) change with the incidence angle of the illuminating beam.
- Volume and surface diffuser. Many cases, like paper, fabrics, paints, involve both scattering by the surface and by the volume.
- Scattering by reflection or by transmission.

1.3 – A special case for an introductory investigation: a set of point sources with random phases

a) Description of the model:



The model of the glass plate with tiny holes in an opaque overcoat (the overcoat is not shown).

We shall consider a set of source points M_j uniformly distributed over a pupil p_o in a plane Π, all coherently emitting spherical waves at wavelength λ. Their phases φ_j are randomly distributed over $[0; 2\pi[$ and independent of each other. One way to imagine the situation is to consider a monochromatic plane wave normally illuminating an plane parallel plate of glass. The glass has been antireflection coated on one side, and coated with a thin perfectly absorbing layer on the other side. Tiny holes have been bored through the absorbing layer with a random thickness whose probability density function extends over much more than $\frac{\lambda}{n-1}$, where n is the index of the glass..

We are interested with light scattered at point $P(\mathbf{r},z)$ in a plane parallel to the glass plate at distance z from plane Π. Denoting by a_o the common modulus of the amplitude emerging from individual holes and reaching plane z, and neglecting the small variations in the modulus due to the distance in the denominator of the standard expression of a spherical wave, we obtain:

$$a(\mathbf{r},z)=\sum_j a_o \exp i\left(\varphi_j+2\pi\frac{M_jP}{\lambda}\right) \quad (1).$$

b) First order statistics:

Using the Central Limit Theorem, a is a Gaussian random process. Because of the pdf of the phases φ_j, it has zero mean and obeys circular statistics (see demonstration in appendix 1). An easy derivation leads to the irradiance probability density function:

$$p_{\mathcal{E}}(e)=\frac{1}{E(\mathcal{E})}\exp\left(\frac{-e}{E(\mathcal{E})}\right) \quad (2).$$

Here, $\mathcal{E}$ stands for the irradiance. That pdf is called a Rayleigh distribution. The “signal to noise” ratio, defined as $\frac{E(\mathcal{E})}{\sigma_{\mathcal{E}}}$, is unity (which is not much!)

c) Second order statistics:

So far, we did not make use of the pupil shape and size. Indeed, the pupil affects the speckle pattern correlation, which is to say the average size of speckle grains. It is measured by the irradiance covariance between two points in the speckle pattern in the observation plane z.

To start with, we calculate the complex amplitude autocovariance, which is this case is identical to its autocorrelation because its statistical expectation is zero:

$$E\left[a^*(\mathbf{r},z)a(\mathbf{r}',z)\right]=\sum_{\text{trous}\, i,j}|a_o|^2\, E\left(\mathrm{e}^{i(\phi_i-\phi_j)}\right)\exp 2i\pi\frac{M_iP-M_jP'}{\lambda}. \quad (3)$$

Since the point sources are independent, the crossed terms vanish. We now move from a continuous sum over the individual point sources to an integral over the diffuser surface and assume that the probability density for the position of the source points M_j is uniform over the pupil. In the Fresnel approximation for the expansion of MP, we find:

$$E\left[a^*(\mathbf{r},z)a(\mathbf{r}',z)\right]=|a_o'|^2\exp i\pi\frac{\mathbf{r}'^2-\mathbf{r}^2}{\lambda z}\int_{pupille}\exp-2i\pi\frac{\mathbf{r_M}\bullet(\mathbf{r}'-\mathbf{r})}{\lambda z}d^2\mathbf{r_M}, \quad (4)$$

with $\mathbf{r},z$ and $\mathbf{r}',z$ the coordinates of P and P’, $\mathbf{r_M},0$ those of M, and where boldface lower case letters designate two-dimensional vectors. Notation a'_o differs from a_o for the sake of homogeneity when moving from a continuous sum to an integral. Whence, if p is the pupil function:

$$E\left[a^*(\mathbf{r},z)a(\mathbf{r}',z)\right]=|a_o'|^2\exp i\pi\frac{\mathbf{r}'^2-\mathbf{r}^2}{\lambda z}\,\tilde{p}\left(\frac{\mathbf{r}'-\mathbf{r}}{\lambda z}\right). \quad (5)$$

The speckle amplitude autocorrelation is therefore seen to be identical to the Fraunhofer diffraction pattern by diffuser the pupil: the larger the illuminated area, the smaller the grains, and conversely. If the illuminated zone is an ellipse with x the longer axis, the grains are elliptical with x the shorter axis, etc..

From the Central Limit Theorem and the Gaussian Moments theorem (Chap I, I.2.e), the irradiance autocovariance is deduced (beware that since the irradiance mathematical expectation is not zero, the irradiance autocorrelation and autocovariance differ by the product of the mathematical expectations):

$$E\left[\mathcal{E}(\mathbf{r},z)\mathcal{E}(\mathbf{r}',z)\right]-E\left[\mathcal{E}(\mathbf{r},z)\right]E\left[\mathcal{E}(\mathbf{r}',z)\right]=$$

$$E\left[a(\mathbf{r},z)a^*(\mathbf{r},z)a(\mathbf{r}',z)a^*(\mathbf{r}',z)\right]-E\left[|a(\mathbf{r},z)|^2\right]E\left[|a(\mathbf{r}',z)|^2\right]=|a_o'|^4\left|\tilde{p}\left(\frac{\mathbf{r}'-\mathbf{r}}{\lambda z}\right)\right|^2. \quad (6)$$

Speckle is therefore a correlated random phenomenon. Its covariance bears information on the pupil function rather than on the scattering phenomenon itself. Since it is correlated, even though the diffuser was "white" in the sense that the individual scatterers M_j were modelled as just "points" in a mathematical sense, speckle is not white noise: its spectral density is equal to the pupil autocorrelation, as we shall see in more detail below. In the next section, we shall investigate the effect of the diffuser being non-white, i.e. having a finite autocovariance width.

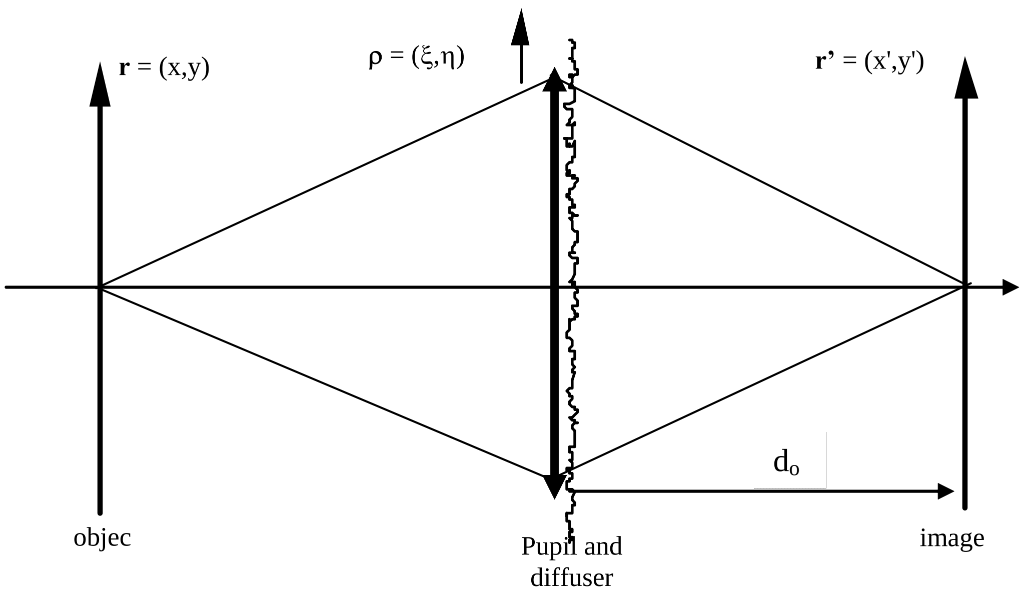
II – FOURIER SPECKLE:

This section extends the special case considered in subsection 1.3 to an arbitrary diffuser limited by a pupil and illuminated by a coherent wave. With the exception of image-plane speckle (section IV below), that is the typical configuration of all speckle situations, with possible complications arising from such effects as diffuser thickness and partial coherence.

2.1 – Description of the configuration:

We shall investigate a more general case than the previous one, but retain the assumption of a thin diffuser, i.e. a diffuser described by a transmittance function that does not depend on the particular illumination. For the sake of specificity, we consider the following imaging setup, where a thin diffuser has been placed in the pupil. Magnification has been set to -1, which is a rather special case and slightly simplifies the calculations but does not affect any meaningful aspect of speckle statistics. Coherent imaging is analysed according to the theory of Fourier Optics reviewed in the first few lectures of this Statistical Optics course: aside from spherical phase factors, the image results from a convolution between the complex amplitude at the exit of the object plane and the system coherent impulse response, which is the complex amplitude of the pupil Fraunhofer diffraction pattern. Therefore, we only need to investigate the latter. Here, the pupil affects a thin, stigmatic lens. It can be decomposed into two physically quite different components:

- the standard, non scattering, diaphragm, typically circular, that limits the lens area,
- the diffuser, which randomly perturbs the phase and modulus of the wave inside the standard pupil. The diffuser is assumed thin, located in immediate contact with the standard pupil, either just in front or just in the back of it (it does not matter as long as the lens and the diffuser are both thin). The diffuser is assumed to be first order and second order stationary in space in the sense of covariance (see Chap I), which intuitively means that there is no visible variation of the statistics over the pupil, like darker or lighter or rougher spots. The diffuser has a "grain", defined in general as the quadratic width of its autocovariance function (see Appendix 2 for the definition of the quadratic width of a function). The grain is assumed very small compared to the standard pupil dimensions.



This distinction between two components allows physically and analytically distinguishing between

- a non-random, non-stationary function with a bounded support, the standard pupil, which mostly consists of low spatial frequencies,
- and a random, stationary function, the diffuser, that can be imagined as extending to infinity. The figure below sketches the imaging situation considered.

Compared to subsection 1.3, two essential changes have occurred:

- the diffuser grain size influence on the speckle pattern can be investigated,

- we have not generally assumed the diffuser to have zero mathematical expectation, therefore we can consider both “weak diffusers” and “strong diffusers”. The model of a weak diffuser is useful, for example, when considering residual scattering by a polished surface. Standard speckle, nevertheless, is “fully developed”: that is the case of a strong diffuser. We shall find it again in slightly different conditions in the next section, devoted to the effect of atmospheric turbulence on imaging in astronomy

Note: here, we consider fully spatially and temporally coherent illumination. Extending this theory to partial coherence is nevertheless relatively easy: the source is decomposed into quasi-monochromatic, spatially coherent elements (“point sources” for short) and the irradiance of individual elements is integrated over all frequencies and all point sources. Indeed, the spatially incoherent case involves the convolution between the object plane luminance and the incoherent impulse response of the system, modulus squared of the coherent impulse response that plays the major role below.

2.2 – Notations:

The pupil function is therefore denoted

$$p(\boldsymbol{\rho}) = p_o(\boldsymbol{\rho}) t_d(\boldsymbol{\rho}). \quad (7)$$

p_o, the “standard pupil”, is the pupil function of a perfect lens with no diffuser present. For numerical applications, we shall consider it to be a disk of radius a. t_d is the diffuser complex amplitude transmission. It exists as such only for thin diffusers. The coherent impulse response reads:

$$CIR(\mathbf{r}') = \frac{1}{\lambda^2 {d_o}^2} \tilde{p}\left(\frac{\mathbf{r}'}{\lambda d_o}\right). \quad (8)$$

We shall investigate its first and second order statistics. The CIR is defined in the image space, it therefore depends on spatial variable $\mathbf{r}'$. In what follows, for simplicity, we shall omit the constant $\lambda^2 {d_o}^2$ in the denominator and consider the CIR to be a function of the reduced variable $\boldsymbol{\omega} = \frac{\mathbf{r}'}{\lambda d_o}$, homogeneous to a spatial frequency.

t_d has in general a non-zero mathematical expectation, denoted by t_o, and which is independent of space because the diffuser is assumed to be first order stationary. We shall denote by $d(\boldsymbol{\rho})$ the zero-mean component:

$$d(\boldsymbol{\rho}) = t_d(\boldsymbol{\rho}) - t_o. \quad (9)$$

A diffuser that has $t_o = 0$ is defined as a “strong diffuser”. For a strong diffuser, the phase must be variable. A typical (but not necessary) case is uniformly distributed phase over $[0; 2\pi[$, which hints at the fact that the random optical path difference through the diffuser fluctuates by several wavelengths. Fourier transforming Eqn (8) yields:

$$\tilde{p}(\boldsymbol{\omega}) = \tilde{p}_o(\boldsymbol{\omega}) t_o + \tilde{p}_o(\boldsymbol{\omega}) * \tilde{d}(\boldsymbol{\omega}). \quad (10)$$

We are interested in the three following basic statistics of the CIR: mathematical expectation, variance and covariance. The mathematical expectation of irradiance and of the incoherent impulse response (IIR) will follow.

2.3 – Statistics of the scattered impulse response:

a) Mathematical expectation:

The mathematical expectation of the CIR complex amplitude is straightforwardly seen to be

$$E(\tilde{p}(\boldsymbol{\omega})) = t_o \tilde{p}_o(\boldsymbol{\omega}). \quad (11)$$

That means that the complex amplitude of the perfect, non scattering, “standard pupil” persists in the image unless the diffuser is “strong”. A weak diffuser yields a non diffuse perfect image surrounded by a speckle pattern, often perceived as a scattered halo.

Outside the diffraction pattern of the perfect pupil – the Airy spot for a circular pupil – the complex amplitude in the CIR has zero mathematical expectation. That does not imply that there is no light, it only means that the phase of the scattered light is distributed in such a way that the mathematical expectation vanishes.

b) Variance:

Eqns (10) and (11) together imply

$$\sigma_{\tilde{p}}^2 = E\left[\left|\tilde{p}(\boldsymbol{\omega}) - \langle \tilde{p}(\boldsymbol{\omega}) \rangle\right|^2\right] = E\left[\left|\tilde{p}_o(\boldsymbol{\omega}) * \tilde{d}(\boldsymbol{\omega})\right|^2\right] \quad (12)$$

We shall see below that this expression can be simplified, but under the present form, it is already clear that light extends far beyond the Airy pattern: the scattered component d sends diffracted light on a wide area outside the

Airy pattern and practically sets the spatial extent of the convolution product in Eqn (12). Compared to section I, the speckle is seen to have a non stationary distribution: $\sigma_{\tilde{p}}^2$ is not constant over the image plane. Instead, its distribution depends on the spatial frequencies present in random process d.

c) Autocovariance:

The speckle amplitude autocovariance is a second order statistics that involves two points in the observation plane, with reduced coordinates $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$. It contains the essential information of how similar two speckle amplitudes are as a function of the distance between the two points:

$$C_{\tilde{p}\tilde{p}}(\boldsymbol{\omega},\boldsymbol{\omega}') = E\left\{\left[\tilde{p}(\boldsymbol{\omega}) - \left\langle \tilde{p}(\boldsymbol{\omega})\right\rangle\right]^* \left[\tilde{p}(\boldsymbol{\omega}') - \left\langle \tilde{p}(\boldsymbol{\omega}')\right\rangle\right]\right\}. \quad (13a)$$

That expression becomes using Eqns (10) and (11):

$$C_{\tilde{p}\tilde{p}}(\boldsymbol{\omega},\boldsymbol{\omega}') = E\left\{\left[\left(\tilde{p}_o * \tilde{d}\right)^*(\boldsymbol{\omega})\right]\left[\left(\tilde{p}_o * \tilde{d}\right)(\boldsymbol{\omega}')\right]\right\}. \quad (13b)$$

Evaluating Eqn (13b) is central to the speckle phenomenon. This can be best done using the following assumptions:

- the autocovariance C_{dd} of the scattering part of the diffuser complex amplitude transmittance is stationary,
- its width g, the diffuser "grain", is small compared to the pupil
- and the pupil is uniform, i.e. function p_o takes on only values 1 and 0.

After the development outlined in Appendix 3, the result is:

$$C_{\tilde{p}\tilde{p}}(\boldsymbol{\omega},\boldsymbol{\omega}') = \tilde{C}_{dd}(\boldsymbol{\omega})\,\tilde{p}_o(\boldsymbol{\omega}-\boldsymbol{\omega}'). \quad (14)$$

It is appropriate to mention that, as opposed to the diffuser itself and to the case of the oversimplified model of subsection 1.3, is not stationary. Under that form, which nevertheless is remarkably simple, the speckle autocovariance comprises one stationary term, which is the Fraunhofer diffraction pattern of the pupil – e.g., for a disk-shaped pupil, the Airy pattern – and one non stationary term, the diffuser spectral density, which according to the second assumption above is slowly varying compared to the stationary term. The stationary term is indicative of the average <u>speckle</u> grain size, while the non stationary term, whose width is inversely proportional to the average <u>diffuser</u> grain size, determines the spatial extent of the speckle pattern in the CIR.

d) Average irradiance of the speckle impulse response:

From Eqn (14), he speckle pattern irradiance autocovariance is derived just like in subsection 1.3 using the Gaussian moment theorem to order 4. Here, we shall only express the mathematical expression of the speckle irradiance at one point $\boldsymbol{\omega}$:

$$\begin{aligned} E\left[\mathcal{E}(\boldsymbol{\omega})\right] &= E\left(\left|\tilde{p}(\boldsymbol{\omega})\right|^2\right) = \left|t_o\tilde{p}_o(\boldsymbol{\omega})\right|^2 + \sigma_{\tilde{p}}^2(\boldsymbol{\omega}) \\ &= \left|t_o\tilde{p}_o(\boldsymbol{\omega})\right|^2 + S\tilde{C}_{dd}(\boldsymbol{\omega}) \end{aligned}. \quad (15)$$

In the last line, the pupil area S, which is identical to $\tilde{p}_o(\mathbf{0})$, was introduced, and the conclusion is that

- an attenuated version of the direct image exists as long as t_o does not vanish
- a halo, usually fairly broad, surrounds the direct image. Its shape depends only on the second order statistics of the diffuser transmittance. The speckle grain <u>statistical average</u> irradiance is given by that halo.

Further development along these lines shows that the irradiance autocovariance is just the squared modulus of the amplitude autocovariance.

<u>2.4 – Some immediate applications:</u>

a) Which information about the diffuser is contained in the speckle pattern:

The issue of whether "all" information about the diffuser statistics was available from the speckle statistics was initially much debated and still occasionally gives rise to some confusion. It is nevertheless obvious that the speckle observed in the Fresnel or, as in the present case, the Fraunhofer diffraction region of the diffuser cannot contain information about spatial frequencies higher than 1/λ. The latter are contained in the evanescent field in the immediate vicinity of the diffuser and are available only in the optical "near field", accessible to "near field optical microscopes", usually nanometre size tips scanning the surface under investigation at nanometre distances. If a diffuser has a grain which is coarse compared to the wavelength, the evanescent part is in fact automatically very small or non existent.

The Fresnel transform, which accounts for light propagation in the Fresnel approximation, is reversible. Therefore, the complex amplitude of a speckle field contains all information about the scattering wavefront where it originates front – or at least, as just mentioned, about its propagating part, as opposed to its evanescent part. On can for example record the hologram of that speckle pattern and thereby reconstruct that wavefront. However, the wavefront does not immediately carry information about the diffuser itself. If the latter is available and if the evanescent part of the field does not play a significant role, one may in principle, by reconstructing the hologram and with due consideration of fine alignment and stability constraints, invert the scattering process by illuminating the diffuser backwards and retrieve the original illuminating beam on the other side of the diffuser: that is the principle of "phase conjugation".

The information content in the speckle pattern irradiance, on the other hand, is much less. In addition, speckle statistics, as opposed to the details of the speckle pattern, carry only a limited amount of information on the diffuser statistics. As has been seen, the grain shape and size depend on the illuminating pupil, not on the diffuser. The higher order statistics are derived from the Gaussian moment theorem according to the central limit theorem: therefore, they carry strictly no information about the diffuser statistics. The only meaningful information is the mathematical expectation of the irradiance, which consists of the central peak (if any) that indicates the average transmittance t_o and the diffuse halo profile which is the spectral density of the scattered part d.

b) Residual diffusion by polished surfaces:

A surface is never perfectly polished: there always remains some level of roughness, be it only at atomic scale. In the current state of the art, the best "super-polish" techniques are used to limit scattering by high precision mirrors. These can be used in the domain of soft X rays, around a wavelength of 10 nm, or for very high quality factor resonators such as those used in gravitational antenna interferometers (Virgo in Europe and its partners in other continents). Returning to visible wavelengths, consider the lens of subsection 2.1 again. If its roughness has a low slope, the analysis in this section is applicable and the lens is in fact a weak diffuser with a transmittance

$$t_d(\boldsymbol{\rho}) = \exp\left(i2\pi\frac{(n-1)h(\boldsymbol{\rho})}{\lambda}\right), \tag{16}$$

Where h is the algebraic height fluctuation with respect to the ideal surface shape. h is small compared to the wavelengths since we are considering the lens as a weak diffuser, therefore it makes sense to expand Eqn (16) to second order in h/λ:

$$t_o = 1 - \frac{2\pi^2(n-1)^2\sigma_h^2}{\lambda^2} = 1 - \frac{\sigma_\varphi^2}{2} \text{ with } \varphi = \frac{2\pi(n-1)h}{\lambda}. \tag{17}$$

Using Eqn 15, it is then straightforwardly seen that a glass surface with a roughness on the order of λ/10 produces a scattered speckle halo power of a few percent of the sharp image.

III – IMAGE OF A FINE GRAINED DIFFUSER

3.1 – Image of a fine grained diffuser under coherent, centred illumination:

This section investigates the speckle formed when observing not the Fraunhofer diffraction pattern of the diffuser, or its Fresnel diffraction pattern, which is similar, as can be seen from subsection 1.3, but its image. We keep most notations of section II and consider the following setup.

On this figure, the dotted lines are just meant to indicate the conjugation between the diffuser and the image plane. The image statistics are strongly influenced by the illumination conditions. In this section, a temporally coherent illumination with vacuum wavelength λ is assumed. A condenser lens, not figured, is used to implement so-called Köhler illumination conditions, where the source, assumed to be plane, is conjugated with the pupil plane by the condenser. In this subsection only, we consider illumination by a point source located on the optical axis, with its image at the centre O of the pupil. Thick dotted lines are not meaningful at the present stage, they are for future use. The distance between the object and the lens is assumed to be equal to that between the lens and the image, namely d_o.

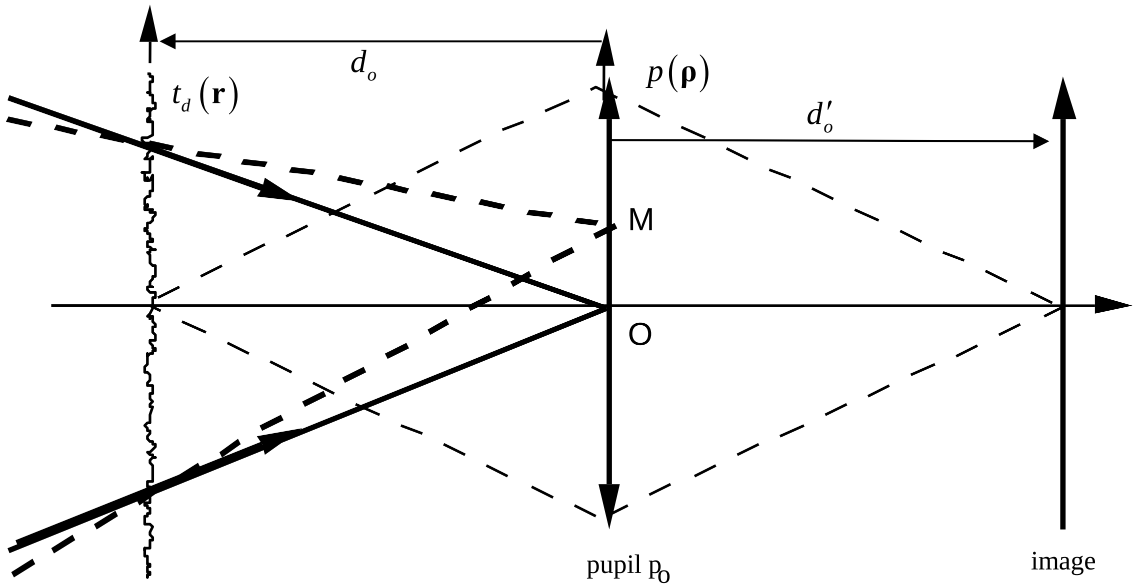
As previously, the diffuser is statistically described by a stationary average t_o and a scattering component d with zero average and stationary correlation, but this time, the proper variable for random process d is $\mathbf{r}(x,y)$, the object plane 2-D vector, rather than the pupil plane variable $\boldsymbol{\rho}$.

$$t(\mathbf{r}) = t_o + d(\mathbf{r}) \tag{20}$$

$$E\left[d(\mathbf{r})^* d(\mathbf{r}')\right] = C_{dd}(\mathbf{r}'-\mathbf{r}). \tag{21}$$

Using the Fourier Optics course on coherent spatial frequency filtering, the image plane complex amplitude, up to a complex constant factor, is expressed in the following simple way, with CIR the coherent impulse response:

$$t'_d(\mathbf{r}') = t'(\mathbf{r}') * CIR(\mathbf{r}') = \frac{1}{\lambda^2 d_o'^2} \int [t_o + d(\mathbf{r})]\, \tilde{p}_o\left(\frac{\mathbf{r}' - \mathbf{r}}{\lambda d'_o}\right) d\mathbf{r} \qquad (22)$$



In that expression, use has been made of the coherent impulse response expression, like in section 2. Notations t' and t'_d are defined in the Fourier Optics course and d' is defined such that $t' = t_o + d'$, whence the notation $C_{d'd'}$, which is just a scaled version of C_{dd}. We now seek the mathematical expectation and the covariance of quantity t'_d. The derivation is similar to that in section 2.

$$E[t'_d(\mathbf{r}')] = t_o p_o(\mathbf{0}) \qquad (23)$$

In that equation, the presence of the pupil transmission value at its centre reflects the fact that the illuminating beam converges at that point. Covariance then boils down to:

$$C_{t'_d t'_d}(\mathbf{r}', \mathbf{r}'_1) = \frac{1}{\lambda^4 d_o'^4} \iint C_{d'd'}(\mathbf{r}_1 - \mathbf{r})\, \tilde{p}_0^*\left(\frac{\mathbf{r}' - \mathbf{r}}{\lambda d'_o}\right) \tilde{p}_o\left(\frac{\mathbf{r}'_1 - \mathbf{r}_1}{\lambda d'_o}\right) d\vec{r}\, d\vec{r}_1 . \qquad (24)$$

The triple product integral of Eqn (24) cannot be further simplified except under specific assumption. Here we shall consider the assumption of “microscopic” correlation of the diffuser, which is a stronger version of the “fine grain” assumption used in section 2. Here, the C_{dd} autocovariance width will not only be considered much smaller than the pupil support but even much smaller than the coherent impulse response width. Therefore, the individual grains in the diffuser are not resolved by the optical imaging system, and there are even many grains per impulse response area. In that case, the integral over $\vec{r}$ in Eqn (24) covers the whole area of function C_{dd} while the impulse response still essentially keeps the value that it takes on at the centre, whence:

$$C_{t'_d t'_d}(\mathbf{r}' - \mathbf{r}'_1) = \frac{\tilde{C}_{dd}(\mathbf{0})}{\lambda^4 d_o'^4} \int \tilde{p}_0^*\left(\frac{\mathbf{r}' - \mathbf{r}_1}{\lambda d'_o}\right) \tilde{p}_o\left(\frac{\mathbf{r}'_1 - \mathbf{r}_1}{\lambda d'_o}\right) d\mathbf{r}_1 . \qquad (25)$$

Going to the Fourier space, and using the fact that the pupil function takes on only values 0 and 1, the following quite simple result is found:

$$C_{t'_d t'_d}(\mathbf{r}' - \mathbf{r}'_1) = \frac{\tilde{C}_{d'd'}(\mathbf{0})}{\lambda^2 d_o'^2}\, \tilde{p}_o\left(\frac{\mathbf{r}'_1 - \mathbf{r}'}{\lambda d'_o}\right). \qquad (26).$$

One important conclusion is the same as in the Fourier speckle case of section 2, namely, that speckle autocovariance has the shape of the impulse response of the illuminating pupil, but in this case the pupil considered is the imaging pupil rather than the support of the diffuser. The complex amplitude statistics are complex Gaussian according to the central limit theorem because the number of diffuser grains whose coherent impulse responses overlap at one given point of the image is large. It can in fact be circular or not, depending on the object phase statistics – the reason for the difference with section 2 here depends on which phase factors are involved in the equations.

The mathematical expectation of irradiance can then be derived. If A is the radius of the pupil, the result is:

$$E[\mathcal{E}(\vec{r}')] = \left|E[t'_d(\mathbf{r}')]\right|^2 + C_{UU}(\mathbf{0}) = t_o^2 p_o(\mathbf{0}) + \tilde{C}_{d'd'}(\mathbf{0}) \frac{\pi A^2}{\lambda^2 d_o'^2} . \qquad (27)$$

3.2 – Image of a fine grain diffuser under non-centred coherent illumination:

We now consider the case where the illuminating beam converges at point M $(\boldsymbol{\rho}_o)$ of the pupil, as indicated on the figure above by the thick dotted lines. Together with its inclined illuminating beam, the object now takes on the form $\left[t_o + d(\mathbf{r})\right]\exp\dfrac{-2i\pi\boldsymbol{\rho}_o\cdot\mathbf{r}}{\lambda d_o}$. Eqn (23) becomes:

$$E\left[t'_d(\mathbf{r}')\right] = t_o p_o(\boldsymbol{\rho}_o)\exp\left[-2i\pi\frac{\boldsymbol{\rho}_o\cdot\mathbf{r}'}{\lambda d'_o}\right] \qquad (23')$$

and the two terms of Eqn (25) undergo the same change, so that Eqn (26) is unchanged:

$$C_{t'_d t'_d}(\mathbf{r}', \mathbf{r}'_1) = \frac{\tilde{C}_{d'd'}(\mathbf{0})}{\lambda^2 d'^2_o}\,\tilde{p}_o\left(\frac{\mathbf{r}'_1 - \mathbf{r}'}{\lambda d'_o}\right) \qquad (26')$$

3.3 – Image of a fine grain diffuser under spatially incoherent illumination :

If the source still has an essentially monochromatic spectrum, but is this time extended, the irradiances due to the various points of the source add up and the mathematical expectation of the image irradiance is the integral over the source, i.e. over variable $(\boldsymbol{\rho}_o)$, of those individual contributions. Several interesting effects result concerning speckle because the two terms of Eqn (27) do not follow the same behaviour as the source extent is increased: the first term is proportional to the source area as long as point M falls inside the pupil, and then remains constant if the source image extends outside the pupil. The other term still keeps increasing, although not proportionally to the source size.

3.4 – Applications:

Three applications can be mentioned here.

- "Subjective" speckle that moves with the eye and is formed on the retina when observing a diffuser illuminated by a laser.
- The behaviour of an overhead projector used to image a rough paper on a screen, with all the light scattered by the paper roughness lost outside the imaging pupil, so that the paper may appear bright when viewed directly and dark when imaged through the projector.
- Photographic effects of historical interest: on the one hand, the grainy appearance of an image is not the exact reproduction of the grain on the photograph itself if the coherent impulse response is smaller than the grain's geometrical image. Under that "fine grain" assumption, the grain noise observed by a detector varies as the square root of the detector area if the detector size itself is larger than the coherent impulse response. The signal to noise ratio then likewise increases as that square root. This effect is known as "Sellwyn's law". The "density", which in photography is defined as the ratio $D = -\log_{10}\dfrac{\text{image illumination with photo absent}}{\text{image illumination with photo present}}$, is not an intrinsic property of the photograph as some scientists who were designing "densitometer" thought in the early times of photography, but depends on the relative aperture of the imaging lens and the condenser in a way that depends on the photographic grain statistics: that is the Callier effect.

APPENDICES

Appendix 1: complex amplitude statistics for a mask with small holes (subsection 1.3)

Starting from the equation

$$a\left(\mathbf{r}_p, z\right) = \sum_j a_o \exp i\left(\varphi_j + 2i\pi \frac{M_j P}{\lambda}\right) = a_o\left(a_r + ia_i\right)$$

with $a_r = \sum_j \cos\left(\varphi_j + 2i\pi \frac{M_j P}{\lambda}\right) = \sum_j \cos\phi_j$ and $a_i = \sum_j \sin\phi_j$. ϕ_j is an evenly distributed random variable over $[0;2\pi[$ just as φ_j , whence

$$E\left(\cos\phi_j\right) = 0 \text{ , } E\left(\cos^2\phi_j\right) = \frac{1}{2} \text{, hence } \sigma_{\cos\phi_j} = 1/\sqrt{2} \text{ ,}$$

A similar result applies for $\sin\phi_j$ and $E\left(\sin\phi_j \cos\phi_j\right) = 0$.

Since a_r is the sum of a large number of independent random variables obeying the same low, the central limit theorem applies and yields that

$$\frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\cos\phi_j - E\left(\cos\phi_j\right)}{\sigma_{\cos\phi_j}} \text{ and } \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \frac{\sin\phi_j - E\left(\sin\phi_j\right)}{\sigma_{\cos\phi_j}}$$

tend for infinite N towards a centred Gaussian random variable with unit variance. In practice, a few tens of holes are enough to guarantee the validity of the central limit theorem. $a\left(\mathbf{r}_p, z\right)$ is therefore the sum of two independent random variables with the same variance: it is a circular Gaussian random variable, with probability density function

$$\frac{1}{\pi N a_0^2} \exp\left(\frac{-|a|^2}{a_0^2 N^2}\right).$$

Appendix 2: the quadratic width of a function

Not every function can be said to have a "width", but generally bell-shaped functions with a bounded support or decreasing fast at infinity can be considered to have a width. There exist several possible definitions for such width of a function. Two fairly simple and intuitive definitions are the distance between the centre and the first zero and the half-maximum width, however they are not easy to manipulate algebraically. A more practical definition is the quadratic width. In this appendix, we review its definition.

Let f be a square-integrable (i.e. "$\mathcal{L}^2$") complex-valued function of real variable, such that its derivative f' and its product with the identity function, x f, also be square-integrable. Then, the "centre" of function f is defined by $t_1 = \frac{\int t|f(t)|^2 dt}{\int |f(t)|^2 dt}$. Its quadratic width is

$$L_{qf} = \left(\frac{\int (t - t_1)^2 |f(t)|^2 dt}{\int |f(t)|^2 dt}\right)^{\frac{1}{2}}.$$

From the Parseval-Plancherel theorem, the quadratic width of a function and of its Fourier transform are related by $L_{qf} L_{q\tilde{f}} \geq \frac{1}{4\pi}$. That expression is also the mathematical basis for the so-called "Heisenberg uncertainty principle" – which does not mean that it is identical to that principle, as is sometimes claimed (the Heisenberg principle is a physical phenomenon, not a mathematical result!) For slightly more complete explanations, one may refer to Appendix 8 of M. Born & E. Wolf, Principles of Optics, Pergamon Press (2000 edition).

Appendix 3: calculating an autocovariance

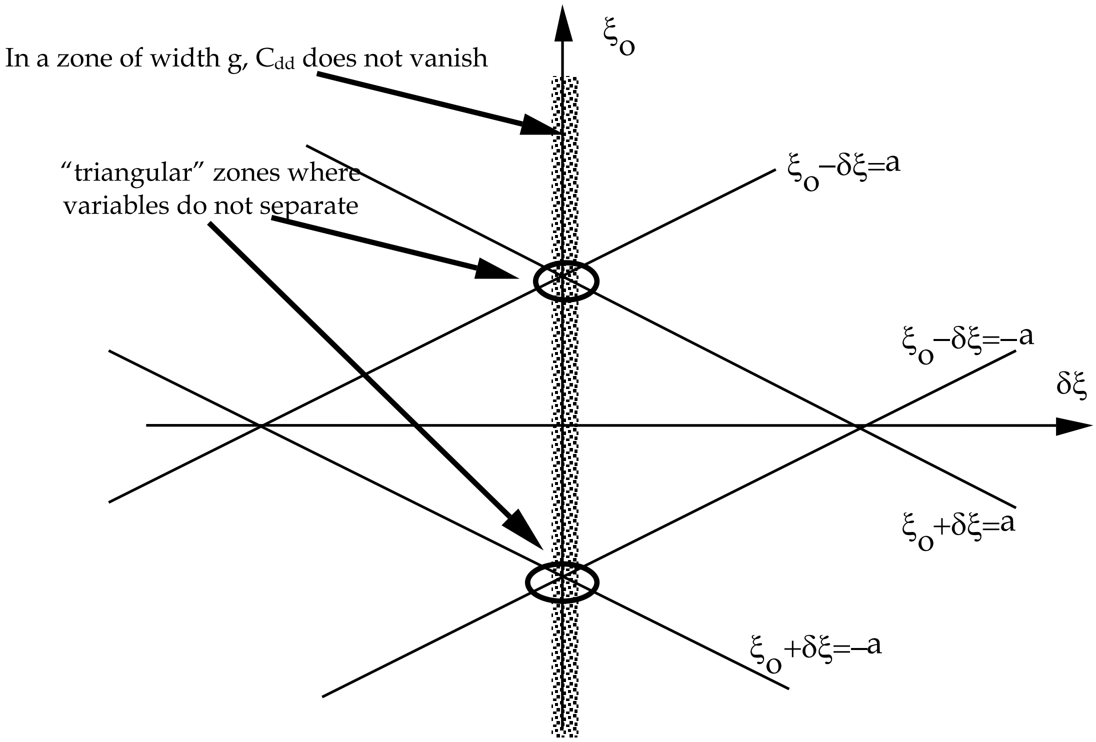
Starting from Eqn (13b), the definition of the convolution of functions is expanded, and then the Fourier transforms are also expressed from their definitions:

$$C_{\tilde{p}\tilde{p}}(\boldsymbol{\omega},\boldsymbol{\omega}') = E\left[\iint \tilde{p}_o^*(\boldsymbol{\omega}_1)\tilde{d}^*(\boldsymbol{\omega}-\boldsymbol{\omega}_1)\tilde{p}_o(\boldsymbol{\omega}_1')\tilde{d}(\boldsymbol{\omega}'-\boldsymbol{\omega}_1')d\boldsymbol{\omega}_1\,d\boldsymbol{\omega}_1'\right]$$
$$= \iint\iint \tilde{p}_o^*(\boldsymbol{\omega}_1)\tilde{p}_o(\boldsymbol{\omega}_1')E\left[d^*(\boldsymbol{\rho})d(\boldsymbol{\rho}')\right]\exp 2i\pi\left[\boldsymbol{\rho}\cdot(\boldsymbol{\omega}-\boldsymbol{\omega}_1)-\boldsymbol{\rho}'\cdot(\boldsymbol{\omega}'-\boldsymbol{\omega}_1')\right]d\boldsymbol{\omega}_1\,d\boldsymbol{\omega}_1'\,d\boldsymbol{\rho}\,d\boldsymbol{\rho}' . \quad \text{(A3.1)}$$

The inverse Fourier transforms of the pupil and its conjugate appear. We next use the following change of variables, which has unit jacobian:

$$(\boldsymbol{\rho},\boldsymbol{\rho}')\mapsto(\boldsymbol{\rho}_o,\delta\boldsymbol{\rho}) \text{ with } \boldsymbol{\rho}=\boldsymbol{\rho}_o-\frac{\delta\boldsymbol{\rho}}{2},\quad \boldsymbol{\rho}'=\boldsymbol{\rho}_o+\frac{\delta\boldsymbol{\rho}}{2}. \quad \text{(A3.2)}$$

$\delta\boldsymbol{\rho}$ is the unique variable in the stationary autocovariance of d[1].



$$C_{\tilde{p}\tilde{p}}(\boldsymbol{\omega},\boldsymbol{\omega}) = \iint p_o^*\left(\boldsymbol{\rho}_o-\frac{\delta\boldsymbol{\rho}}{2}\right)p_o\left(\boldsymbol{\rho}_o+\frac{\delta\boldsymbol{\rho}}{2}\right)C_{dd}(\delta\boldsymbol{\rho})\exp 2i\pi\left[\boldsymbol{\rho}_o\cdot(\boldsymbol{\omega}-\boldsymbol{\omega}')-\delta\boldsymbol{\rho}\cdot\frac{\boldsymbol{\omega}+\boldsymbol{\omega}'}{2}\right]d\boldsymbol{\rho}_o\,d\delta\boldsymbol{\rho} \quad \text{(A3.3)}$$

To proceed further from Eqn (A3.3), the assumption that the grain is small with respect to the pupil is required. We now examine the support of the three functions in the integrand (aside the exponential) in the ξ_o, $\delta\xi$ plane along the section $\eta_o = 0,\ \delta\eta = 0$ (see Figure). For the figure, a circular pupil with radius a has been assumed, so that in the section plane, function $p_o\left(\boldsymbol{\rho}-\frac{\delta\boldsymbol{\rho}}{2}\right)$ takes on value unity in a band between the two straight lines of equations $\xi_o-\frac{\delta\xi}{2}=\pm a$ and vanishes outside these bands. The same assumption holds for function $p_o\left(\boldsymbol{\rho}+\frac{\delta\boldsymbol{\rho}}{2}\right)$ with the band between lines $\xi_o+\frac{\delta\xi}{2}=\pm a$. The autocovariance C_{dd}, on the other hand, depends in that section plane only from $\delta\xi$ and extends physically around the centre only on a width of the order of one "grain" g, by assumption much smaller than a, outside which it vanishes. Outside the two small triangular regions highlighted on the figure, one can separate the variables in the integral of (A3.3): when integrating over $\delta\xi$: in the region where C_{dd} takes on non negligibly small values, the two pupil functions essentially do not change: they are equal to unity inside the above-mentioned bands,

[1] Since d has zero mathematical expectation, its autocovariance and its autocorrelation are identical.

and therefore outside the region $-a \le \xi_o \le a$ ending with the triangles, and vanish outside, but they do not depend on $\delta\xi$ and therefore take on the same value as for $\delta\xi=0$. Eqn (A3.3) then becomes:

$$C_{\tilde{p}\tilde{p}}\left(\boldsymbol{\omega},\boldsymbol{\omega}'\right) = \tilde{C}_{dd}\left(\frac{\boldsymbol{\omega}+\boldsymbol{\omega}'}{2}\right)\int p_o^*\left(\boldsymbol{\rho}_o\right) p_o\left(\boldsymbol{\rho}_o\right)\exp -2i\pi\left[\boldsymbol{\rho}_o \cdot \left(\boldsymbol{\omega}'-\boldsymbol{\omega}\right)\right] d\boldsymbol{\rho}_o \,, \tag{A3.4}$$

which can be further simplified by considering that p_o takes on only values 0 and 1:

$$C_{\tilde{p}\tilde{p}}\left(\boldsymbol{\omega},\boldsymbol{\omega}'\right) = \tilde{C}_{dd}\left(\frac{\boldsymbol{\omega}+\boldsymbol{\omega}'}{2}\right)\tilde{p}_o\left(\boldsymbol{\omega}'-\boldsymbol{\omega}\right). \tag{A3.5}$$

Finally, the latter expression can be simplified by using the theorem regarding the width of Fourier transform pairs : the pupil Fourier transform is narrow compared to the variations of the diffuser spectral density $\tilde{C}_{dd}$, so that at the scale of that latter function variables $\boldsymbol{\omega}$ and $\boldsymbol{\omega}'$ are essentially equal when the product in Eqn (A3.5) does not vanish. Whence Eqn (14).