Chapter 4

Theory of Optical waveguides

In Chapter 2, we have reviewed the key results of waveguide theory, particularly with respect to the various optical modes that can exist in the waveguide. A comparison has been made between the physical-optic approach and the ray-optic approach in describing light propagation in a waveguide. In this chapter, the electromagnetic wave theory of the physical-optic approach is developed in detail. Emphasis is placed on the two basic waveguide geometries that are used most often in optical integrated circuits, the planar waveguide and the rectangular waveguide.

4.1 Lorentz reciprocity theorem for z-invariant system

We begin by considering a guide with an arbitrary cross-section as illustrated in Fig. 4.1. The axis of the guide will be taken as the $z$-axis. We plan to derive general properties for the modes supported by the waveguide. The modes, defined by the field distributions $E_n(r)$ and $H_n(r)$ and labeled by the subscript $n$, are solutions of Maxwell’s equations in the absence of any source

$$\nabla \times E_n(r) = i\omega \mu \epsilon_n(r), \quad \text{(4.1a)}$$

$$\nabla \times H_n(r) = -i\omega \epsilon \mu E_n(r), \quad \text{(4.1b)}$$

see Chapter 1 for details. We have assumed that the permeability is everywhere that of vacuum, and $\epsilon$ is the effective permittivity tensor. The latter is a function of the $x$ and $y$ coordinate only (i.e. $\partial \epsilon / \partial z = 0$). Since the whole dielectric structure is homogeneous along the $z$ axis (since there is no source too), the solutions of Maxwell’s equations can be written

$$E_n(r) = e_n(x,y) \exp[i(\beta_n z - \omega t)], \quad \text{(4.2a)}$$

$$H_n(r) = h_n(x,y) \exp[i(\beta_n z - \omega t)], \quad \text{(4.2b)}$$

where $\beta_n$ is a propagation constant to be determined. The basic problem is that of finding the solution of Eqs. (4.1a) and (4.1b) with the fields given by Eqs. (4.2a) and (4.2b) subject to the continuity conditions on the tangential components of the fields at the dielectric interfaces (tangential $E$ and $H$ are continuous) and the outgoing wave conditions at infinity. Given a refractive index profile $n^2(r) = \epsilon(r)$, there is in general an infinite number of eigenvalues $\beta_n$,
corresponding to an infinite number of modes that are eigenvectors of the differential operators of Eqs. (4.1a) and (4.1b). By construction, z-invariant waveguides possess a mirror-symmetry for transverse planes perpendicular to the z-axis. For any given forward-propagating mode with a \( \exp(i\beta z) \) dependence, this guarantees the existence of a backward-propagating with an \( \exp(-i\beta z) \) dependence.

Among the infinite number of eigenvalues \( \beta_n \), normally only a finite number of these modes are confined near the core and propagate freely (absorption or roughness are neglected) along the waveguide.

One of the necessary conditions for a guided mode is that there is no transverse (along the \( x \) and \( y \) directions) flow of energy (no leakage). This requires that the fields fall off exponentially outside the guide structure. Consequently, see Chapter 2, \( (\omega/c)^2 n^2 - \beta_n^2 \) must be negative in the region far away from the guiding region (core). In other words, where \( \beta \) is the propagation constant \( \beta \) of a truly-guided (confined) mode must be such that

\[
\beta_n^2 > (\omega/c)^2 n^2(x),
\]

where \( n(x) \) is the refractive index at infinity, \( (x^2+y^2)^{1/2} \to \infty \).

**Figure 4-1.** Sketch of a z-invariant waveguide with an arbitrary cross section. The orange parallelepiped formed by two cross-section planes, \( z = z_1 \) and \( z = z_2 \) is used to apply Lorentz reciprocity.

### 4.2 Mode orthogonality

To derive the orthogonality of (normal) modes, we consider the closed surface formed by two cross-sections, \( z = z_1 \) and \( z = z_2 \), and we apply the generalized Lorentz reciprocity theorem, Eq. (1.40a), taking for solution 1, the normal mode labeled \( m E_1 = E_m, H_1 = H_m \), that is solution of
Maxwell’s equations at a frequency $\omega_1 = \omega$ for $\varepsilon_1 = \varepsilon(r)$ without sources ($J_1 = 0$) and for solution 2, another normal mode labeled $n$, $E_2 = E_n, H_2 = H_n$, that is also solution of Maxwell’s equations at the same frequency $\omega_2 = \omega$ for the same relative-permittivity distribution $\varepsilon_2 = \varepsilon(r)$. Again since the second solution is a mode, there is no source and $J_2 = 0$. We obtain
\[
\iint_{\Sigma} (E_n \times H_m - E_m \times H_n) \cdot d\Sigma = \iint_{\Sigma} \varepsilon_0 \left( E_n \cdot \varepsilon E_m - E_m \cdot \varepsilon E_n \right) + \mu_0 \left( H_n \cdot H_m - H_m \cdot H_n \right) d\Omega.
\] (4.4)

Since we have applied the Lorentz reciprocity theorem, it is assumed that the materials are reciprocal ($\varepsilon = \varepsilon^T$); thus $E_n \cdot \varepsilon E_m = E_m \cdot \varepsilon E_n$ and the term on the right side of Eq. (4.4) is null
\[
\int\int_{\Sigma} (E_n \times H_m - E_m \times H_n) \cdot d\Sigma = 0.
\] (4.5)

Substituting Eqs. (4.2a) and (4.2b) into Eq. (4.5) and assuming that the transverse cross-sections of the box at $z = z_1$ and $z = z_2$ ($z_1 \neq z_2$) have an infinite spatial extend, the contribution of the four other boundaries vanishes. This is obvious provided that the two normal modes are truly guided, since their fields fall off exponentially outside the guide structure\(^1\). Thus Eq. (4.5) becomes
\[
\left[ \exp(i(\beta_m + \beta_n)z_2) - \exp(i(\beta_m + \beta_n)z_1) \right] \int\int_{C} (e_n \times h_m - e_m \times h_n) \cdot \hat{z} \, dx \, dy = 0,
\] (4.6)

where $C$ represents any transverse cross-sections of the waveguide and $\hat{z}$ is the vector parallel to the $z$ axis with $|\hat{z}| = 1$. Note that the integral does not depend on the $z$-coordinate since $e_n, h_n$ do not depend on $z$. The first term on the left side of Eq. (4.6) is null if and only if $\beta_m + \beta_n = 0$, we finally get
\[
\langle \Psi_n | \Psi_m \rangle = \int\int_{C} (e_n \times h_m - e_m \times h_n) \cdot \hat{z} \, dx \, dy = \delta_{n,m},
\] (4.7)

where $|\Psi_n \rangle$ is a bra-ket notation to design the mode $n$ formed by the 6-component vector $[E_n, H_n]$ and $\delta_{q,p}$ is equal to 1 for $p = q$, and 0 otherwise. Note that $\langle \Psi_n | \Psi_m \rangle$ does not denote the classical scalar product between column and row vectors, but a more general bilinear form.

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\(^1\) We will admit without demonstration that the contribution is also null for the radiative normal modes, even so their field might diverge as $(x^2+y^2)^{1/2} \to \infty$. The mathematics related to the derivation of the orthogonality of radiation normal modes are not obvious, and the orthogonality condition has been established for a number of limited cases, in particular for slab waveguides that are invariant in two directions.
Equation (4.6) represents the orthogonality condition of two normal modes existing at a given frequency. The condition is somewhat unusual since a normal mode is orthogonal to all other normal modes (including itself), except potentially to normal modes with an opposite propagation constant. This mode (there might be several in case of degeneracy), which corresponds to the same mode that is propagating in the opposite direction, always exists as will be shown hereafter.

The relationship (4.7) is very general; the sole assumption is that the materials are reciprocal. When the material are lossless (no absorption, $\epsilon = \epsilon^*$) it is possible to derive another orthogonality condition in the usual sense of Poynting $E \times H^*$ products.

### 4.3 Mode orthogonality for lossless waveguides

The absorptionless or weak absorption case ($\epsilon = \epsilon^*$) is of high importance in practice, since virtually all waveguides used in integrated optics are lossless. If $|\Psi_n\rangle$ is a normal mode, it obeys Maxwell’s equations

\[
\nabla \times E_n(r) = i\omega \mu_0 H_n(r),
\]

\[
\nabla \times H_n(r) = -i\omega \epsilon_0 E_n(r).
\]

Since $\epsilon = \epsilon^*$, it is easily shown that the 6-component vector $[E_n^*, -H_n^*]$ obeys the same Maxwell’s equations, proving that if

\[
E_n(r) = e_n(x, y) \exp[i(\beta_n z - \omega t)],
\]

\[
H_n(r) = h_n(x, y) \exp[i(\beta_n z - \omega t)],
\]

then

\[
E_n^*(r) = e_n^*(x, y) \exp[i(-\beta_n z - \omega t)],
\]

\[
-H_n^*(r) = -h_n^*(x, y) \exp[i(-\beta_n z - \omega t)],
\]

is also a mode (with an opposite propagation constant $-\beta_n$) of the waveguide.

The derivation of the mode orthogonality is strictly identical to that performed for the general case, except that we take the normal mode of Eqs. (4.10), $E_2 = E_n^*$, $H_2 = -H_n^*$, as the solution 2. We obtain
The first term on the left side of Eq. (4.11) being not null, except if \( \beta_m = \beta_n \), we finally get

\[
\langle \psi_n | \psi_m \rangle_p = \iint_C \text{Re} \left( \mathbf{e}_n \times \mathbf{h}_m^* \cdot \hat{z} \right) dxdy = \delta_{n,m},
\]

where the subscript \( P \) is introduce for differentiation with Eq. (4.7) and to emphasize the fact that the orthogonality is achieved in the sense of the Poynting vector.

Equation (4.12) has very important consequences in practice for lossless waveguides. To illustrate that, let us consider the scattering problem of Fig. 4.2, where light is focused onto the input facet of an optical waveguide. In the waveguide section for \( z > 0 \), the field can be expanded into the set of all normal modes (this is known as the completeness principle\(^2\) for the normal modes) including radiation modes, modes propagating towards positive \( z \)-values, or negatives ones, and we may write

\[
|\psi\rangle = \sum_m a_m \psi_m,
\]

for \( z > 0 \). Possibly the discrete sum could be an integral over a continuum. The \( a_m \)'s are the expansion coefficients that are in general unknown and \( |\psi\rangle \) is the 6-component vector formed by the electromagnetic field \( [\mathbf{E}, \mathbf{H}] \) that is solution of the scattering problem. In any cross-section \( C \) of the waveguide it is interesting to calculate the power flux flowing through it,

\[
F = \frac{1}{2} \iint_C \text{Re} \left( \mathbf{E} \times \mathbf{H}^* \cdot \hat{z} \right) dxdy,
\]

and using elementary algebra, we have

\[
F = \frac{1}{2} \iint_C \text{Re} \left[ \left( \sum_m a_m \mathbf{E}_m \right) \times \left( \sum_n a_n \mathbf{H}_n \right)^* \right] \cdot \hat{z} dxdy,
\]

\[
F = \frac{1}{2} \iint_C \text{Re} \left[ \left( \sum_m \sum_n a_m a_n^* \mathbf{E}_m \times \mathbf{H}_n^* \right) \right] \cdot \hat{z} dxdy,
\]

and by using Eq. (4.12)

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\(^2\) The principle of completeness of normal modes stipulates that the set of radiation and guided modes is complete. This has been proven for some simple structures like lossless slab waveguides see, H. Sagan, *Boundary and eigenvalue problems in mathematical physics*, Dover Publications., NY (1989). However, a formal completeness proof is lacking in general especially for absorbing materials. Anyway, like in many textbooks, we will admit that Eq. (4.13) is always valid in any \( z \)-invariant section of a waveguide.
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\[ F = \sum_m |a_m|^2 \iint \frac{1}{2} \text{Re} \left( E_m \times H_m^* \right) \hat{z} \, dx \, dy. \]  

(4.16)

implying that the power carried through any cross-section \( C \) of the waveguide can be decomposed as a sum of powers carried by the individual modes of the waveguide. This important result places mode concepts at the heart of the understanding of energy transport in guided waves. Equation (4.12) that is valid only for non lossy materials is by far preferable to Eq. (4.7), which is always valid but not intuitive.

Figure 4-2. Light injection into a waveguide \((z > 0)\). \( C \) represents an arbitrary cross section.

4.4 Existence of pairs of counter-propagating modes

Directly from Maxwell’s equations, it is easily shown that if the materials are reciprocal, one can associate to any mode \((e_x, e_y, e_z, h_x, h_y, h_z) \exp(i\beta z - \omega t)\) a counter-propagating mode of the form \((e_x, e_y, -e_z, -h_x, -h_y, h_z) \exp(-i\beta z - \omega t)\). This is a consequence of the symmetry with respect to any transverse plane of the waveguide. An interested reader may directly check this property by writing the Maxwell’s curl equations in Cartesian (or cylindrical) coordinates for instance, followed with a few elementary algebraic operations. Thus the set of normal modes of optical waveguides are composed of pairs with opposite propagation constants, \(\beta\) and \(-\beta\), and it is sufficient to know the mode profile of one element of the pair to know the other. Figure 4.3 shows a symmetric dispersion diagram that illustrates our purpose. For a lossless waveguide \(\varepsilon = \varepsilon^*\), we additionally know that if \((e_x, e_y, e_z, h_x, h_y, h_z) \exp(i\beta z - \omega t)\) is a mode, \((e_x, e_y, e_z, -h_x, -h_y, -h_z) \exp(-i\beta z - \omega t)\) is also a mode, see Eqs. (4.10) and (4.9). Thus provided that there is no mode degeneracy, we have

\[ e_x = e_x^*, e_y = e_y^*, e_z = -e_z^*, h_x = h_x^*, h_y = h_y^*, h_z = -h_z^*, \]  

(4.17)

showing that the field components are either real or imaginary for lossless waveguides.

In general \(\beta\) might be real, imaginary or complex, and classifying the modes according to
the sign of the real or imaginary parts of $\beta$ is difficult, especially for highly absorbing materials. However hopefully in the present context, in the case of lossless waveguides, the propagation constants $\beta$ of normal modes are purely real or imaginary and modes propagating towards the positive $z$-direction correspond to positive values of $\text{Re}(\beta)$ (guided modes) and $\text{Im}(\beta)$ (radiation modes exponentially damping as they propagate).

![Dispersion diagram](image)

**Figure 4-3.** Dispersion diagram showing the symmetry with respect to the $y$-axis. The effective index $n_{\text{eff}}$ given simply reads as $\frac{\beta}{(\omega/c)}$ and the group velocity $v_g$ is given by $c$ times the slope of the dispersion curve $d(\omega/c)/d\beta$. At point $A$, both $v_g$ and $n_{\text{eff}}$ are negative, whereas they are positive at $A'$.

### 4.5 Group velocity

The group velocity of a wave is the velocity with which the overall shape of the wave amplitudes — known as the envelope of the wave — propagates through space. For example, imagine what happens if a light pulse is propagating into a fiber with a negligible deformation. If you observe the pulse at position $z$ at time $t$, then you will observe it at position $z'$ at time $t'$, provided that $(z'-z)/(t'-t)$ be equal to the group velocity, i.e. to the averaged speed of the different temporal harmonics that compose the pulse.

In general, the velocity at which a pulse of light propagates through a medium (group velocity) is given by $v_g = c \left( n + \omega \frac{dn}{d\omega} \right)^{-1}$ where $c$ is the velocity of light in a vacuum, $\omega$ is the light’s angular central frequency of the pulse and $n$ is the refractive index of the medium. The group velocity depends not only on the refractive index $n$, but also on the dispersion (i.e., $dn/d\omega$). For a pulse propagating into a mode of a waveguide, it is easy to see that the refractive index $n$
has to be replaced by the effective index \( n_{\text{eff}} = \beta/k_0 \) of the mode. Thus the group velocity reads as

\[
v_g = c \left( n_{\text{eff}} + \omega \frac{dn_{\text{eff}}}{d\omega} \right)^{-1} = \frac{d\omega}{d\beta},
\]

and can be directly read on the dispersion diagram, see Fig. 4.3. It is noteworthy that now, the group index depends not only on the material dispersion (\( n_{\text{eff}} \) changes with the wavelength because the refractive index \( n(r) \) changes), but also on the geometrical dispersion (even if \( n(r) \) is constant, \( n_{\text{eff}} \) generally varies as shown in the previous chapters). Except near mode cutoff, the material and geometrical dispersions are small, and the group velocity is close to \( c/n_{\text{eff}} \) for most optical waveguides.

The group velocity is also often thought of as the velocity at which energy or information is conveyed along a wave. Actually this is correct in most cases, and the group velocity can be thought of as the signal velocity of the waveform (exceptions occur in the presence of absorption), as we shall see now.

To show that the group and energy velocities are the same for optical waveguides, we consider the closed surface formed by two cross-sections, \( z = z_1 \) and \( z = z_2 \), and we apply the generalized Lorentz reciprocity theorem, Eq. (1.40a), taking for solution 1, the normal mode \( |\Psi\rangle = |e_1, h_1\rangle \exp(i\beta(\omega_1)z) \), which is solution of Maxwell’s equations at a frequency \( \omega_1 \) for \( \varepsilon_1 = \varepsilon(r, \omega_1) \) without sources (\( J_1 = 0 \)) and for solution 2, the corresponding counter-propagating normal mode, \( |\Psi\rangle = |e_2, h_2\rangle \exp(-i\beta(\omega_2)z) \), which is also solution of Maxwell’s equations for the same waveguide but at another frequency \( \omega_2 \) with \( \varepsilon_2 = \varepsilon(r, \omega_2) \). Possibly, \( \varepsilon_1 \) and \( \varepsilon_2 \) are different because of material dispersion. Writing down all steps of the derivation we obtain,

\[
\frac{\text{exp}(i(\beta_1 - \beta_2)z_2) - \text{exp}(i(\beta_1 - \beta_2)z_1)}{\beta_1 - \beta_2} \int e_1 \times h_1 \cdot \hat{z} \, dx \, dy = \int e_2 \times h_2 \cdot \hat{z} \, dx \, dy,
\]

where we assume as before that only the integral over the cross-section planes, \( z = z_1 \) and \( z = z_2 \), contributes to the integral on the left-hand side of the equation. For the sake of simplicity, we also note \( \beta(\omega_2) \) by \( \beta_2 \) and \( \beta(\omega_1) \) by \( \beta_1 \). As \( \omega_1 \rightarrow \omega_2 \equiv \omega \), \( \beta_1 \rightarrow \beta_2 \equiv \beta \), the transverse components of \( e_1 \), denoted \( e_{1,T} \), tend towards \( e_{2,T} \equiv e_T \) (the subscript \( T \) being either \( x \) or \( y \)) and those of \( h_1 \) towards \( -h_{2,T} \equiv -h_T \), whereas \( e_{1,z} \rightarrow -e_{2,z} \equiv -e_z \) and \( h_{1,z} \rightarrow h_{2,z} \equiv h_z \). Thus with a Taylor expansion, one obtains
In this form, Eq. (4.19) (which is perhaps false, checking is needed) is difficult to interpret, but let us consider the asymptotic case for which the absorption vanishes. Then because \( e_T = e_T^*, e_z = -e_z^*, h_T = h_T^*, \) and \( h_z = -h_z^* \), see Eq. (4.17), we obtain

\[
\frac{1}{2} \int_{\Sigma} \left( e_T \times h_T^* \right) \cdot \hat{z} \, d\Sigma \\
= \frac{d\omega}{d\beta} \frac{1}{4} \int_{\Sigma} \left[ \epsilon_0 e_T \cdot \frac{\partial (\epsilon \omega \mu)}{\partial \omega} e_T - \epsilon_0 e_z \cdot \frac{\partial (\epsilon \omega \mu)}{\partial \omega} e_z + \mu_0 h_T \cdot \frac{\partial (\omega \mu \epsilon)}{\partial \omega} h_T - \mu_0 h_z \cdot \frac{\partial (\omega \mu \epsilon \beta)}{\partial \omega} h_z \right] d\xi \, d\eta \, d\phi \, d\chi. \tag{4.21}
\]

which simply read as

\[
\frac{1}{2} \int_{\Sigma} \left( e_T \times h_T^* \right) \cdot \hat{z} \, d\Sigma = \frac{d\omega}{d\beta} \frac{1}{4} \int_{\Sigma} \left[ \epsilon_0 e \cdot \frac{\partial (\epsilon \omega \mu)}{\partial \omega} e^* \right] d\xi \, d\eta \, d\phi \, d\chi. \tag{4.22}
\]

because \( e_T \) and \( h_T \) are both real function of \( x \) and \( y \). As shown in [Lan76], the right term under the integral is remnant of the electromagnetic energy density in dispersive and weakly-lossy materials. Strictly speaking, if one neglects absorption loss, one should neglect material dispersion, and because \( \frac{\partial (\epsilon \omega \mu)}{\partial \omega} = \epsilon \) and \( \frac{\partial (\omega \mu \epsilon \beta)}{\partial \omega} = \mu \) and we then get

\[
\frac{1}{2} \int_{\Sigma} \left( e_T \times h_T^* \right) \cdot \hat{z} \, d\Sigma = \frac{d\omega}{d\beta} \frac{1}{4} \int_{\Sigma} \left[ \epsilon_0 e \cdot \epsilon e^* + \mu_0 h \cdot \mu h^* \right] d\xi \, d\eta \, d\phi \, d\chi. \tag{4.22}
\]

Equation (4.22), in which we recognize the Poynting vector on the left side and the electromagnetic energy density on the right side, simply reads as a conservation law that interpret the group velocity \( d\omega/d\beta \) as an energy velocity.

References