Optical models for color reproduction

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1. INTRODUCTION

Coloring a surface to form an image is a very ancient activity which is based on a simple principle: the deposition of coloring layers on a reflecting support. In painting or analog photography, these coloring layers are continuous. They cover the entire surface and more or less absorb light depending on their thickness and dye concentration. The obtained colors are called ‘continuous tone’ or ‘contone’. In printing, the coloring layers are discontinuous. The inks have fixed thickness and colorant concentration; the different tones are obtained by varying their surface coverage ratio yielding the ‘halftones’ colors presented in detail in the previous papers. Historically, the color rendering of images was controlled by the painter, the photographer or the printer, i.e. by a specialist who somehow acquired his expertise in the selection of the materials and the control of the coloration process. The new printing technologies have increased the image reproduction quality and the ubiquity of color images in everyday’s life. They have also provided the possibility for non-expert consumers to print themselves at home thanks to fully automated printing processes. In the absence of a color expert, the printer needs to be calibrated by the constructor. This requires to relate color rendering to technical parameters (dye concentration, ink thickness, etc.) according to a scientific approach relying on physical measurements. The task of scientists in this domain is double: understanding the physical phenomena being at the origin of the color rendering, and predicting the color rendering for given printing specifications. These two topics are the subject of the following papers.

Before entering into the physical characterization of the printed colors, it should be recalled that color is not a physical quantity but a physiological sensation. This sensation is the response of our visual system to a light signal striking the retina. During the 20th Century, scientists managed to elaborate a mathematical description of the color sensation, and to connect it with the spectrum of the light received by the retina. However, the study of color and the study of light are two scientific domains based on very different concepts, called respectively colorimetry and optics. Light can be characterized by its energy, its speed, its wavelength, its direction, its polarization, but it would be erroneous to say that it "has a color". Likewise, it is simplistic to say that a surface "has a color". We should rather say that it has the aptitude to reflect a fraction of the ambient light depending on its spectrum attenuation capacities, which creates a light signal that human brain perceives as a color. A complete description of the print color rendering should therefore rely on physical as well as perceptual analysis. However, we generally assume that these two analyses can be treated separately, i.e. the light signal issued from the surface is fully characterized by the physics,
and the interpretation of this light signal in terms of color is described by colorimetry. In this paper, we will focus on the physical analysis.

Optics is the scientific study of light and its interaction with matter. It covers a wide range of phenomena and applications. We focus here on the basic notions necessary to understand surface color prediction models. We first recall briefly what is light and what kind of light is considered in color reproduction. Then, we introduce radiometry, the branch of optics that deals with light measurement, as well as models for absorption, reflection and refraction by a surface and scattering.

2. LIGHT

According to the Commission Internationale de l’Eclairage (CIE), light is the generic name for the electromagnetic radiations visible to the human eye [1]. As every electromagnetic radiation, it can be considered as a wave phenomenon as well as a collection of particles called photons, propagating at a speed \( c \approx 2.998 \times 10^8 \text{ m.s}^{-1} \) in vacuum [2]. The simplest emission mechanism of light is due to isolated atoms which emit a photon when transiting from a high energy level to a lower energy level. Since the energy of atoms is quantized, only a finite number of possible energies can be given to the emitted photon. To each photon energy there corresponds a monochromatic wave characterized by its oscillation frequency \( \nu \) or its wavelength \( \lambda = \frac{c}{\nu} \). The vibration frequency is proportional to the photon energy \( e \) according to the relation \( e = h\nu \), where \( h \approx 6.626 \times 10^{-34} \text{ J.s} \) is the Planck constant. The sensibility of the human eye to light is significant for wavelengths between 400 and 700 nanometers (nm), with a maximum around 555 nm (photopic vision) or around 501 nanometers in dark context (scotopic vision). However, the notion of light can be extended to infrared (IR) and ultraviolet (UV) radiations, having respectively longer and shorter wavelengths but similar physical properties.

2.1. Wave optics and polarization

According to the wave model, light is composed of an electric field and a magnetic field oscillating in phase, perpendicular to each other and both perpendicular to the propagation direction. The two fields are modeled as three-dimensional vectors, respectively \( \mathbf{E} \) and \( \mathbf{B} \), being functions of time and position. Maxwell’s equations describe their variation in time and space according to the electrical properties of the propagation medium. When the medium is not vacuum, the propagation speed \( v \) is slower than \( c \). The ratio \( c/v \), called refractive index, characterizes the optical properties of the medium.

The electric and magnetic fields oscillate in the plane orthogonal to the propagation direction. Polarization denotes the time-dependent orientation of the electric field \( \mathbf{E} \) (thereby of the
magnetic field which is perpendicular to it) in this plane [3]. A convenient way to describe polarization is to project vector $\mathbf{E}$ on two orthogonal axes of the plane. One obtains two components $E_x$ and $E_y$ being periodical functions of time with identical period, whose phase difference indicates the polarization. If $E_x$ and $E_y$ oscillate in phase, $\mathbf{E}$ oscillates according to a straight line and polarization is said to be linear (see Figure 1). If the phase difference between $E_x$ and $E_y$ is $\pm \pi/2$, $\mathbf{E}$ draws a circle and polarization is said to be circular. Elliptic polarization corresponds to the other phase differences. Polarization is generally modified when light interacts with matter, for example when it is reflected or refracted at the interface between media with different refractive indices.

![Diagram of linear and circular polarization](image)

Figure 1. Propagation of the electric field for linear and circular polarizations of light.

**2.2. Natural light**

The light emitted by the sun and most common light sources is composed of many short wave packets independent of each other and having different polarizations. Polarization therefore varies rapidly in a random and irregular manner. Such light is called natural light. It is modelled as the sum of two linearly polarized lights independent of each other, whose respective electric fields oscillate in perpendicular directions. These two polarized lights are generally denoted by the symbols $p$ and $s$, whose meaning will appear clearly in Section 4.2. Their respective powers $\Phi_p$ and $\Phi_s$ determine the degree of polarization (DOP) defined as [4,5]:

$$DOP = \frac{\max \left( \Phi_p, \Phi_s \right) - \min \left( \Phi_p, \Phi_s \right)}{\max \left( \Phi_p, \Phi_s \right) + \min \left( \Phi_p, \Phi_s \right)}$$

(1)
The DOP is 0 when the two components have equal power and 1 when one of the two components is zero, which corresponds respectively to *unpolarized* and *totally polarized natural light*. A DOP value between 0 and 1 indicates that the light is *partially polarized*.

### 2.3. Light ray and geometrical optics

The light ray concept is evident for everyone. It comes from the observation that light propagates along straight lines in homogenous media, e.g. air or clear water. However, it has no physical existence. It is only an approximated model describing the propagation of light when its wave property can be ignored. In practice, the ray concept is sufficient to describe reflections and refractions as well as the trajectory of light beams through optical systems, provided the successive variations of refractive index occur at large distances compared to the wavelength. Light rays are the basis of a domain of optics called *geometrical optics*.

Light rays are independent of each other. This means that there is no *spatial coherence* between them. When two rays are superposed, their energies are simply added without any interference or diffraction phenomenon. According to Huygens’ principle [6], the condition for this independence property is that the cross section of the ray (also called *extent*) is much larger than the wavelength. The principle of Fermat, also known as the *least time principle*, asserts that light follows the quickest optical path between two given points. In a medium of constant refractive index, light propagates at constant speed and the quickest path is a straight line. If the refractive index varies, the quickest path may follow a bended or curved line. The set of points being at the same optical length from a point source is called the *wave surface*. *Malus’ law* asserts that the wave surface is always perpendicular to the ray, even after various reflection or refraction events [6]. In a light pencil, the optical length between two wave surfaces is the same for all rays.

### 2.4. Interaction between light and colored surfaces

The materials used in printing, i.e. papers, plastics and inks, have a very complex structure. The refractive index of the materials, which determines how light propagates in them, varies locally in an irregular manner. The paper fibers and the ink pigments provoke multiple diffraction events which would be impossible to describe all rigorously. However, since heterogeneities are randomly distributed in the media, these diffraction events yield no perceptible colored effect. They simply contribute to a global light diffusion process. Instead of describing the complex, random paths followed by waves in the colored materials, one rather considers average photon transfers such as the transfer from a source to a detector or the transfer from one layer to another. This assumption considerably simplifies light-print interaction models and allows staying in the geometrical optics domain. The measurement of these light quantities and the study of their distribution in space are the aim of radiometry.
3. RADIOMETRY

Radiometry is the science of the measurement of radiations. It comprises the study of radiation emission by sources, detection, reflection or transmission through optical systems, etc. It thus gives rise to a profuse literature (see for example references [5, 7, 8, 9, 10, 11, 12, 13]). Radiometry differs from optics in the sense that it focuses on energy measurement, without having to consider specifically any type of material. However, it is crucial to know the properties of light to perform appropriate measurements and interpret them correctly. Most of the radiometry is based on incoherent radiations and on the geometrical optics of rays. Most of the time, wave phenomena such as diffraction and interferences are ignored. Radiations are measured in terms of absolute power and the measurements are expressed in energy units. The considered type of radiation depends on the source and the detector. In color reproduction, we consider electromagnetic radiations of wavelength comprised between 400 and 750 nm, i.e. visible light. The perception of light by a human observer is studied in a separate discipline called photometry. The radiant power at each wavelength is weighted by a visual sensitivity function that models human brightness sensitivity. The measured quantities are expressed in luminous units called lumen, candela and lux, especially used for the characterization of light sources. The distinction between optical radiometry and photometry is linked to the history of radiation measurement, but they use similar concepts and are almost synonymous today.

The interest of radiometry in appearance assessment is the possibility to quantify the amounts of light being in interaction with the object and their transfer from a source to a detector. The fundamental radiometric quantities describe the geometrical distribution of energy in space and the reflection or transmission properties of objects can be defined as ratios of them, called accordingly reflectance or transmittance. In this section, we introduce the radiometric definitions leading to the reflectance and transmittance concepts and address the main tools for their measurement.

3.1. Geometrical concepts

Describing the transport of light from a source to an object, then from the object to an observer is first a question of geometry. If one defines a light ray as the photons passing through two points \( P_1 \) and \( P_2 \), the ray would contain no photon because the probability for a photon to meet precisely one point or to follow one direction is zero. One should rather consider a small area around each point and a small set of directions. The set of directions is called solid angle, and its coupling with a small area is called geometrical extent.
An infinitesimally small solid angle points at one direction which is generally specified by its polar and azimuth angles \((\theta, \phi)\) in spherical coordinates. Figure 2 shows that it intercepts an area \(x^2 \sin \theta d\theta d\phi\) on the sphere of radius \(x\). The infinitesimal solid angle is therefore

\[
d\omega = \sin \theta d\theta d\phi
\]

Figure 2. Differential solid angle in the direction \((\theta,\phi)\).

The geometrical extent denotes the geometry of a light pencil propagating between two small surface elements, \(ds_1\) and \(ds_2\) (Figure 3).

By assuming that the distance \(x\) between them is sufficiently large, one may consider that the rays received by \(ds_2\) come from one point \(P_1\) on \(ds_1\). Similarly, one may consider that the rays emitted by \(ds_2\) reach one point \(P_2\) on \(ds_2\). The line \((P_1P_2)\) gives the direction of the light ray. It forms an angle \(\theta_1\) with the normal of \(ds_1\) and an angle \(\theta_2\) with the normal of \(ds_2\).

Figure 3. Elementary pencil of light between two small surface elements \(ds_1\) and \(ds_2\).
The solid angle based in $P_1$ subtended by $ds_2$ intercepts an area $dA_2$ on the sphere of radius $x$ centered in $P_1$. It is therefore

$$d\omega_1 = dA_2 / x^2 = ds_2 \cos \theta_2 / x^2$$

(3)

Likewise, the solid angle based in $P_2$ subtended by $ds_1$ is

$$d\omega_2 = dA_1 / x^2 = ds_1 \cos \theta_1 / x^2$$

(4)

The geometrical extent of the light pencil [5], expressed in $m^2.sr$, is defined as

$$d^2G = dA_1d\omega_1 = dA_2d\omega_2 = \frac{dA_1dA_2}{x^2} = \frac{1}{x^2}(ds_1 \cos \theta_1)(ds_2 \cos \theta_2)$$

(5)

3.2. Radiometric quantities

Radiometric rules are based on four fundamental quantities: radiant flux, radiant intensity, irradiance and radiance. *Radiant flux* $\Phi$ (or simply *flux*) is the energy radiated per unit time expressed in watts (W).

*Radiant intensity* $I$ is the density of flux per unit solid angle that is emerging from a point in space and propagating in a specified direction $d\omega$ (expressed in $W.sr^{-1}$)

$$I = \frac{d\Phi}{d\omega}$$

(6)

Intensity is rather used for point sources that cannot be given a well defined area, such as stars in astronomy.

*Irradiance* $E$ is the density of flux per unit area that is incident on a specified point in a specified surface, expressed in $W.m^{-2}$. If one considers a flux $d\Phi$ relatively to a surface element $ds$, the corresponding irradiance is

$$E = \frac{d\Phi}{ds}$$

(7)

Irradiance is a function of position on the surface. *Exitance* $M$ is the equivalent of irradiance when light emerges from the surface instead of being incident.

*Radiance* $L$ is the flux per unit extent that is incident on, passing through or emerging from a specified point in a specified surface in a specified direction, in $W.m^{-2}.sr^{-1}$

$$L = \frac{d^2\Phi}{d^2G}$$

(8)

Radiance is the most suitable radiometric quantity to describe thin light pencils. Its relationship with the geometrical extent provides interesting geometrical properties: if we consider a flux propagating between two small surface elements, we can be certain that the radiance emitted by the one is equal to the radiance received by the other one. This principle,
called the *radiance invariance*, is a direct consequence of equation (5). If one considers a surface element $ds$ and a differential solid angle $d\omega = \sin \theta d\theta d\phi$ oriented at $\theta$ to the normal of $ds$, the differential extent is $dG = ds \cos \theta d\omega$. The radiance is thus expressed as

$$L(\theta, \phi) = \frac{d^2\Phi}{ds \cos \theta d\omega} \quad (9)$$

In equation (9), the ratio $d^2\Phi / ds$ corresponds to the elemental irradiance $dE$ attached to the light pencil. It is related to the radiance by

$$dE(\theta, \phi) = L(\theta, \phi) \cos \theta d\omega \quad (10)$$

### 3.3. Photometric units

Photometry differs from radiometry by the fact the energy is weighted by the spectral sensibility of the human eye [14], usually denoted as $V(\lambda)$, plotted in Figure 4.

![Figure 4. Spectral sensitivity of the human eye in photopic vision.](image)

The flux is called *luminous flux* and is expressed in lumen (lm). The equivalent for irradiance is *illuminance* in the case of an illuminated surfaces or *luminous exitance* in the case of an emitting surface (expressed in lux). The *luminous intensity* is expressed in candela (cd) and the radiance, called *luminance*, is expressed in candela per square meter ($\text{cd.m}^{-2}$).

### 3.4. Lambert’s law

Although the notion of diffuse light is intuitive, it can be given a rigorous meaning thanks to the radiance concept. According to Lambert’s law, a perfectly diffusing surface emits or reflects the same radiance in every direction. It is thus called a *Lambertian surface* or *reflector*.

According to relation (10), the elemental exitance issued from the surface in some direction $(\theta, \phi)$ is

$$dM(\theta, \phi) = L \cos \theta d\omega \quad (11)$$
where \( d\omega = \sin \theta d\theta d\varphi \) is the infinitesimal solid angle and \( L \) is a constant. By summing up the elemental exitance elements over the hemisphere, one obtains the following relation between radiance \( L \) and total exitance \( M \):

\[
M = \int_{\varphi=0}^{\pi} \int_{\theta=0}^{\pi/2} L \cos \theta \sin \theta d\theta d\varphi = \pi L
\]

In practice, many sources are Lambertian and strongly scattering materials such as paper bulk, cotton furniture or milk are Lambertian reflectors.

### 3.5. BRDF

According to Nicodemus [15], the reflection process of light by a surface is embodied in the fundamental equation relating the elemental irradiance \( dE_i \) coming from each direction \((\theta_i, \varphi_i)\) and the radiance \( dL_r(\theta_r, \varphi_r) \) reflected into each direction \((\theta_r, \varphi_r)\)

\[
dL_r(\theta_r, \varphi_r) = f_R(\theta_i, \varphi_i; \theta_r, \varphi_r) dE_i(\theta_i, \varphi_i)
\]

Function \( f_R \) is called bidirectional reflectance distribution function (BRDF). Thanks to the relation (10), it can be defined in terms of the incident radiance \( L_i(\theta_i, \varphi_i) \):

\[
dL_r(\theta_r, \varphi_r) = f_R(\theta_i, \varphi_i; \theta_r, \varphi_r) L_i(\theta_i, \varphi_i) \cos \theta_r d\omega_i
\]

Figure 5. Sections of BRDF in the incidence plane \((\varphi_i = \varphi_r = 0)\), plotted in polar coordinates as a function of \( \theta_i \) of (a) a Lambertian reflector, (b) a smooth surface, (c) a roughened aluminium surface and (d) a glossy paper [16].

In the case of a nonabsorbing Lambertian reflector, the total outgoing exitance \( E_r \) is equal to the incident irradiance. Since the reflected radiance is \( E_r/\pi \) in every direction, the BRDF is a constant equal to \( 1/\pi \). Thus, the spectral BRDF depends only on wavelength and is easier to
measure. In the case of a perfect mirror, the BRDF cannot be defined. A setup for mirror reflectance measurement is described in Ref. [17, p. 54]. Figure 5 shows BRDF sections in the incidence plane of a Lambertian reflector, a smooth surface, roughened aluminium surface [18] and a glossy paint.

BRDF is a function of many parameters: the four angles denoting the incidence and observation directions, and possibly wavelength of light, polarization, position on the surface… It is therefore impossible to plot a full BRDF on 2D graphic. 3D visualization by software is often preferred. However, planar mapping is good alternative for BRDF display [19].

The Lambert azimuthal equal-area projection [20] is especially convenient as it conserves areas: a portion of hemisphere with area $A$ is mapped into a portion of disk with same area $A$. Every point $P$ on the hemisphere of radius 1, specified by its spherical coordinates $(\theta, \phi)$, is mapped to a point $P'$ of polar coordinates $(r, \phi)$ contained within a disk of radius $\sqrt{2}$ tangent to the hemisphere at the North pole N (Figure 6).

The azimuth coordinate $\phi$ is the same in the two coordinate systems. Coordinate $r$ corresponds the distance $NP$:

$$r = 2\sin(\theta/2)$$

![Lambert Projection](image)

Figure 6: Mapping of the hemisphere onto a disk of radius $\sqrt{2}$ according to Lambert azimuthal equal-area projection applied at the North pole N.

Point $P'$ is also specified by the following Cartesian coordinates

$$u = 2\sin(\theta/2)\cos\phi$$
$$v = 2\sin(\theta/2)\sin\phi$$

(15)

Applying this mapping to the BRDF yields a multispectral image containing as many channels as wavelengths. Each pixel of the image corresponds to a same solid angle. Square pixels with size $d$ represent solid angles of $d^2$ steradians.
Spectral BRDF is measured with a *gonio-spectrophotometer*. The incident light, generally brought by an optical fiber, is collimated with an optical system in such manner as to illuminate the sample with well parallel rays. In classical configurations, a rotating arm enables choosing the incidence direction. The reflected light is captured through a very thin solid angle by an optical system located on a second rotating arm. Light is then transferred to a spectrophotometer. Spectral measurements are performed at different detector positions. Although gonio-spectrophotometers can now be found at reasonable prices on the marketplace, their volume and fragility restrict their use for specific applications in laboratory.

### 3.6. Reflectance

The term *reflectance* denotes any ratio of reflected flux to incident flux being relative to a same surface element. Reflectance is therefore a ratio of exitance to irradiance. It is a dimensionless quantity depending on wavelength, direction, polarization and position on the surface. In this paper, we consider isotropic surfaces whose reflection properties are independent of position on average over areas of a few squared millimeters. We also consider natural light in the visible spectral domain. Thus, all radiometric quantities are spectral quantities defined for each of the two polarized components. However, as for the BRDF, dependence on wavelength and polarization are made implicit.

Many different reflectances can be defined in regard to the cone $\Gamma_i$ through which light incomes and the cone $\Gamma_r$ through which reflected light is observed. In the general case, the incident radiance $L_i(\theta_i, \phi_i)$ is a function of direction $(\theta_i, \phi_i)$ and creates, according to equation (10), the elemental irradiance $dE(\theta_i, \phi_i) = L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i$. The total irradiance originating from $\Gamma_i$ is given by the following integral:

$$E_{\Gamma_i} = \int_{(\theta, \phi) \in \Gamma_i} L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i$$  \hspace{1cm} (16)

The contribution of some incident radiance $L_i(\theta_i, \phi_i)$ to one radiance $L_r(\theta_r, \phi_r)$ is specified by the BRDF according to equation (14). The corresponding elemental exitance is

$$dM(\theta_i, \phi_i; \theta_r, \phi_r) = f_R(\theta_i, \phi_i; \theta_r, \phi_r) L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i \cos \theta_r d\omega_r$$  \hspace{1cm} (17)

By summing up the contributions of all radiances contained within $\Gamma_i$, one obtains the elemental exitance in the direction $(\theta_r, \phi_r)$

$$dM(\Gamma_i; \theta_r, \phi_r) = \int_{(\theta, \phi) \in \Gamma_i} f_R(\theta_i, \phi_i; \theta_r, \phi_r) L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i \cos \theta_r d\omega_r$$  \hspace{1cm} (18)

Then, by summing up all the elemental exitances through the observation solid angle $\Gamma_r$, one obtains the total exitance
The ratio of the exitance to the irradiance is the reflectance defined by the cones $\Gamma_i$ and $\Gamma_r$, denoted as $R_{\Gamma_i;\Gamma_r}$:

$$R_{\Gamma_i;\Gamma_r}(\theta_i, \phi_i; \theta_r, \phi_r) = \frac{\int_{(\theta_i,\phi_i) \in \Gamma_i} \int_{(\theta_r,\phi_r) \in \Gamma_r} f_R(\theta_i, \phi_i; \theta_r, \phi_r) L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i \cos \theta_r d\omega_r}{\int_{(\theta_i,\phi_i) \in \Gamma_i} \int_{(\theta_r,\phi_r) \in \Gamma_r} L_i(\theta_i, \phi_i) \cos \theta_i d\omega_i}$$

(20)

Judd [21] then Nicodemus [15] defined nine geometries where each of $\Gamma_i$ and $\Gamma_r$ is either directional, conical or hemispherical. By considering an isotropic reflector (BRDF independent of the incident azimuth angle $\phi_i$) and a Lambertian illumination (constant radiance $L_i$), expression (20) noticeably simplifies. For example, the directional-hemispherical reflectance $R(\theta_i)$, or simply directional reflectance, is defined for directional illumination at $(\theta_i, \phi_i)$ and a hemispherical observation:

$$R(\theta_i) = \int_{\phi_i = 0}^{2\pi} \int_{\theta_i = 0}^{\pi/2} f_R(\theta_i, \phi_i; \theta_r, \phi_r) \cos \theta_i \sin \phi_i d\theta_i d\phi_i$$

(21)

The bi-hemispherical reflectance $r$, or simply Lambertian reflectance is defined for Lambertian illumination through the hemisphere and hemispherical observation. It can be directly related to the directional reflectance [22]:

$$r = \int_{\theta_i = 0}^{\pi/2} R(\theta_i) \sin 2\theta_i d\theta_i$$

(22)

3.7. Reflectance factor

The reflectance measurement relies on two flux measurements: the reflected flux and the incident flux. As most instruments contain one detector which is used to capture the reflected flux, the incident flux cannot be measured directly. It is measured indirectly by using a perfect white diffuser able to reflect the incident light uniformly over the hemisphere without absorbing it. The flux captured by the detector is therefore proportional to the incident flux. The ideal white standard is a perfectly Lambertian, nonabsorbing and diffusing sample [23]. Its reflectance is equal to 1 and its BRDF is 1/$\pi$ for every couple of incidence and reflection directions. In practice, white standards approaching these properties are made of pressed barium sulfate or PTFE (known as Algoflon, Halon or Spectralon). They must be calibrated in terms of the perfectly reflecting diffuser. The object to assess and the perfect diffuser are illuminated and observed with the same geometry. The ratio $\hat{R}$ of the flux $\Phi$ measured from the object to the flux $\Phi_{ref}$ measured from the white diffuser is called reflectance factor [24]:

$$\hat{R} = \frac{\Phi}{\Phi_{ref}}$$

(23)
A alternative definition is sometimes used when radiance measurements are performed: the ratio of radiance $L$ measured from the object to radiance $L_{\text{ref}}$ measured from the white diffuser is thus called *radiance factor* [17].

The reflectance and radiance factors are not rigorously reflectances. They coincide with reflectance in the case of Lambertian reflectors and provide a good approximation for matte papers and other nearly Lambertian reflectors. It is less relevant however for non-Lambertian reflectors such as glossy papers, mirrors or satin paintings. In some cases, the sample reflects more light towards the detector than the perfect diffuser and the reflectance factor overpasses one [25]. This occurs for example with a mirror illuminated by directional flux $\Phi_i$ at angle $\theta_i$ and observed by a radiance detector in the specular direction. The detector captures the flux $\Phi = R\Phi_i$ from the mirror, where $R$ denotes the mirror's angular reflectance at the considered incidence, and the flux $\Phi_{\text{ref}} = \Phi_i / \pi$ from the perfect diffuser. In this configuration, the reflectance factor of the mirror is higher than one at every wavelength where $R(\theta_i) > 1 / \pi$.

### 3.8. Transmittance

All the definitions presented above regarding the reflectance of objects can be transposed to the transmittance. The equivalents for BRDF, reflectance and reflectance factor are respectively *BTDF* (*bi-directional transmittance distribution function*), *transmittance* and *transmittance factor*. This latter is defined in respect to the perfectly nonabsorbing transmitter, which in practice is air.

### 3.9. Spectral radiometry

The previous radiometric quantities have been defined without consideration of wavelength. The spectral distribution of the radiation is described by a *spectral flux* $\Phi_\lambda$, defined as flux per unit wavelength (in W.m$^{-1}$):

$$\Phi_\lambda = \frac{d\Phi}{d\lambda}$$  

(Spectral flux is measured with a spectrophotometer, generally in successive spectral bands. If the wavelength bandwidths $\Delta\lambda$ are small, the measured flux in each bandwidth is $\Phi_\lambda \Delta\lambda$. Over a larger band $[\lambda_1, \lambda_2]$, the measured flux is

$$\Phi_{[\lambda_1, \lambda_2]} = \int_{\lambda_1}^{\lambda_2} \Phi_\lambda d\lambda$$

The spectral resolution of spectrophotometers varies according to the application and the method used to decompose the light spectrum. For color reproduction applications, usual commercial instruments have a resolution comprised between 1 and 10 nanometers. In order to select narrow wavelength bandwidths, the light is decomposed according to a dispersing
prism or a diffraction grating [26]. The location of the photodetector determines the measured wavelength domain. For faster measurements, the different wavelengths may be captured simultaneously by using an array of sensors (diode linear array, CCD linear array, etc.)

One similarly defines spectral intensity $I_\lambda$, spectral irradiance $E_\lambda$, spectral radiance $L_\lambda$.

The spectral reflectance is the ratio of reflected to incident spectral fluxes, both defined in the same small bandwidth $\Delta \lambda$ around the considered wavelength $\lambda$:

$$ R(\lambda) = \frac{\Phi_{\lambda,r}\Delta\lambda}{\Phi_{\lambda,i}\Delta\lambda} $$  

*Spectral BRDF* $f_r(\lambda)$, and *spectral reflectance factor* $\hat{R}_r(\lambda)$, as well as *spectral BTDF* $f_t(\lambda)$, spectral transmittance $T(\lambda)$ and spectral transmittance factor $\hat{T}_t(\lambda)$ are similarly defined. In the following sections, all the radiometric quantities are spectral quantities but in order to simplify the notations, the term spectral and the index $\lambda$ will be omitted.

### 3.10. Light sources and illuminants

The light source is a crucial element for the optical characterization of reflecting objects. Reliable characterization is possible only if the spectrum of the incoming light is non-zero for each wavelength band. The luminous power of the source should also be adapted to the sensitivity of the photosensor. A too weak flux at a given wavelength may decrease the signal-to-noise ratio of the detection system and induce a significant error in the measurement. Oppositely, a too strong flux saturates the photosensors. In the adequate power range, for each spectral band, one can assume that the reflected flux varies linearly with the incident flux (except in the special case of fluorescing objects which are treated in the last section of this paper). The ratio of these fluxes is therefore a constant, independent of the source power, which will be defined as being the *reflectance* of the object. However, although the source has no direct impact on optical properties of the object, its spectral power distribution (SPD) influences the perception of color. In colorimetry, relative SPD is called *illuminant*. Classical color spaces such as the CIELAB color space take it into account in the computation of the tri-chromatic coordinates of the colors. It is possible that two objects with different spectral reflectance $R_1(\lambda)$ and $R_2(\lambda)$ have the same color under the illuminant $I(\lambda)$ and different colors under the illuminant $J(\lambda)$, because the spectral radiances $R_1(\lambda)I(\lambda)$ and $R_2(\lambda)I(\lambda)$ correspond to metameric spectra whereas the spectral radiances $R_1(\lambda)J(\lambda)$ and $R_2(\lambda)J(\lambda)$ do not.

The ideal illuminant for reflectance measurement would be the *equal energy illuminant* $E$ whose relative SPD is uniform over the visible spectrum. However, no natural or artificial lighting has a uniform SPD. In order to assess color rendering for most common lightings, the CIE defined various illuminants [14] inspired of the SPDs of incandescent light (illuminant
A), of daylight (illuminants D) and of fluorescent lightings (illuminants F). Some of them are plotted in Figure 7. The relative SPD of illuminant A is issued from the spectral radiance of a black body at the temperature $T_A = 2848$ K given by Planck's law, normalized to the value 100 at the wavelength $\lambda_0 = 560$ nm:

$$S_A(\lambda) = 100 \left(\frac{\lambda_0}{\lambda}\right)^5 \frac{\exp\left(\frac{hc}{\lambda_0 k T_A}\right)}{\exp\left(\frac{hc}{k k T_A}\right) - 1}$$  \hspace{1cm} (27)

where $c = 2.998 \times 10^8$ m.s$^{-1}$ is the speed of light in vacuum, $h = 6.626 \times 10^{-34}$ J.s is the Planck constant and $k = 1.380 \times 10^{-23}$ J.K$^{-1}$ is the Boltzmann constant.

Figure 7. Spectral distribution power of CIE standard illuminants A, D65, D50 and F11.

The D series of illuminants were constructed by Judd, MacAdam, and Wyszecki to represent natural daylight [27]. The D50 and D65 illuminants are especially used in graphical industry and paper industry. Their spectra, plotted in Figure 7, are known to correspond to horizon daylight and noon daylight spectra. They are easy to characterize mathematically since they may be derived from the linear combination of three spectra. However, they are difficult to produce artificially. Figure 8 shows two examples of light sources considered as D65 illuminants with noticeably different spectra: the light source in the SpectroEye spectrophotometer from X-rite and the ‘Color Control Classic Line’ light table from Just Normlicht. The F series of illuminants represent various types of fluorescent lighting. The ability of real light sources to reproduce the D65 illuminant can be assessed with the CIE metamerism index [28]. The F11 illuminant, plotted in Figure 7, is a narrow triband illuminant consisting of three narrowband emissions in the red, green and blue regions of the visible spectrum, obtained by a composition of rare-earth phosphors.
3.11. Measurement geometries

The reflectance measurement devices designed for color reproduction applications contain either directional or Lambertian white light source and capture the reflected light either in one direction (radiance measurement) or over the hemisphere thanks to an integrating sphere (irradiance measurement). The spectrum of the source generally tends to reproduce the color of a standard illuminant [29], typically the D65 illuminant, despite the difficulty to reproduce reliably the illuminant spectra defined by the CIE with artificial lightings (see Figure 8). Once captured, light is transferred to a spectrophotometer which measures the flux in the different wavelength bands 1, 5 or 10 nm wide. Table 1 presents some geometries recommended by the CIE for reflectance measurement [14].

Table 1. Some of the geometries recommended by the CIE for reflectance measurements

<table>
<thead>
<tr>
<th>Appellation</th>
<th>Illumination</th>
<th>Capture</th>
</tr>
</thead>
<tbody>
<tr>
<td>Diffuse / 8° geometry, specular component included (di:8°)</td>
<td>Diffuse</td>
<td>Radiance detector (8°)</td>
</tr>
<tr>
<td>Diffuse / 8° geometry, specular component excluded (de:8°)</td>
<td>Diffuse</td>
<td>Radiance detector (8°)</td>
</tr>
<tr>
<td>Diffuse / diffuse geometry (d:d)</td>
<td>Diffuse</td>
<td>Integrating sphere</td>
</tr>
<tr>
<td>Alternative diffuse geometry (d:0°)</td>
<td>Diffuse</td>
<td>Radiance detector (0°)</td>
</tr>
<tr>
<td>45° annular / normal geometry (45°a:0°)</td>
<td>Directional</td>
<td>Radiance detector (0°)</td>
</tr>
<tr>
<td>45° directional / normal geometry (45°x:0°)</td>
<td>Directional</td>
<td>Radiance detector (0°)</td>
</tr>
</tbody>
</table>

*Integrating spheres* are spherical cavities internally coated with a powder of nonabsorbing material, e.g. barium sulfate (BaSO₄), behaving as a perfect diffuser [30, 31]. They can be used either to produce a Lambertian illumination or to collect reflected light over the hemisphere. Figure 9 illustrates these two possibilities.
Figure 9. Integrating spheres used in a 0°:d geometry (left) and a d:0° geometry (right).

In the d:0°, the integrating sphere plays the role of diffuser for the illuminating flux. The reflected light is captured at 0° or 8° in respect to the normal of the sample. In the case of diffusing samples having a flat surface, one may want to discard the specular reflection component from the measurement. A hole located in the regular direction in respect to the detector position ensures that the specular reflection component is not captured [17]. A hemispherical-directional reflectance is measured. In the 0°:d geometry, the integrating sphere collects the whole flux reflected by the sample, which is illuminated by a collimated beam. The corresponding reflectance is a directional reflectance, given by equation (21).

In contrast with integrating spheres, radiance detectors capture only a fraction of the flux issued from the specimen. This fraction depends on both area and solid angle of the detector, which are generally unknown. The so-called 45°:0° geometry, widely used in the printing industry, is a bidirectional geometry where light is incident at 45° and a radiance detector captures light at 0° [31]. The sample is illuminated from one or all azimuth directions, yielding respectively the directional and annular variants of the 45°:0° geometry. The annular geometry, illustrated by Figure 10, minimizes texture and directionality whereas the directional geometry tends to enhance them.

Figure 10. 45° annular / normal geometry (45°:a:0°) for reflectance measurements.
Several companies such as X-rite, Datacolor and Konica Minolta have developed spectrophotometers able to measure both reflectance and transmittance. They are typically based on the di:8° and de:8° geometry in reflectance mode, and on the d°:0° geometry in transmittance mode. Note that all these measurement geometries make sense when the sample is diffusing. Using them with nonscattering sample, either in reflectance or transmittance mode, needs some precaution. For example, when measuring the transmittance of a nonscattering filter with a d:0 geometry, only the radiance normal to the sample is captured by the detector (Figure 11).

![Diagram of transmittance measurement](image)

**Figure 11.** Transmittance measurement of a nonscattering filter with a d:0 geometry. Only the radiance incident at 0° is captured and the light coming from other directions is ignored by the detector. The effective measurement geometry is the 0:0 geometry.

The effective measurement geometry is therefore the 0:0 geometry. The incident radiance at 0° is obtained by measurement without the sample. Thus, ratio of measurements with the sample and without it provides the directional transmittance at 0°. We have similar configuration with a mirror using a d:8 geometry in reflectance mode, where the effective geometry is the 8:8 geometry. The incident radiance is obtained by performing a measurement on a reference mirror whose spectral reflectance is perfectly known. As an alternative, it can be obtained from a measurement of a perfect non-absorbing diffuser: the radiance reflected at 8° coincides with the radiance incident at 8° on the mirror ($E/\pi$ in both cases, where $E$ is the total incident irradiance). In the case of weakly scattering samples, however, the amount of incident light being able to reach the detector cannot be certainly known since it depends on the scattering diagram of the medium. The reflectance and reflectance factor concepts have no pertinence any more. Only BRDF and BTDF measurement can provide reliable information on the reflecting and transmitting properties of the sample.
4. REFLECTION AND REFRACTION

Two media of different refractive indices have in common a planar boundary called \textit{interface}. The optical properties of the interface depend on its \textit{relative refractive index}, i.e. the ratio of the refractive indices of the two media. When the interface is flat, each of its faces reflects and transmits unidirectional light into one couple of directions, called \textit{regular} or \textit{specular directions}, attached to the reflected and refracted components. Reflection and refraction play an important role in the interaction of light with printed supports. Everyone has observed the reflection of light by the surface of a glossy photograph. At the other side of the surface, the diffuse light coming from the paper and the inks is also reflected. Thus, light travels several times in average between the paper substrate and the inks before exiting the print definitively.

4.1. Refractive index

The refractive index of a medium is a measure of the propagation properties of light in that medium. It is generally a complex number depending on wavelength:

\[
\hat{n}(\lambda) = n(\lambda) + i\kappa(\lambda)
\]  

The real part \( n(\lambda) \), called \textit{real refractive index}, is related with the light propagation speed. The imaginary part \( \kappa(\lambda) \), called \textit{extinction index}, characterizes absorption by the medium. Table 2 gives the refractive indices of a few common materials.

Ellipsometry is the favourite technique for refractive index measurements. It is based on polarization analysis. The constraint is that the sample must be homogenous, nonscattering and very flat, which makes this technique almost impossible to use with printing materials such as inks and paper. Note that the real and imaginary functions of wavelength are related to each other by the Kramers-Kronig relations [32, 33, 34]. Thus, knowing either the real index or the extinction index over the whole spectrum enables obtaining the other one for any wavelength.

For dielectric materials such as glass, plastic or paper fibres, the attenuation index is low compared to the real index. The refractive index may be considered as being real and absorption is modelled independently by an attenuation factor applied to the ray (\textit{see} the section on Beer’s law). The dependence of the real index on wavelength, at the origin of the \textit{dispersion} phenomenon [33] as well as the chromatic aberrations in optical systems [35], is empirically modelled in the visible wavelength domain by Cauchy’s law [2, 36]:

\[
n(\lambda) = a_1 + \frac{a_2}{\lambda^2}
\]  

where the dimensionless factor \( a_1 \) and the coefficient \( a_2 \) (in m\(^{-2}\)) are to be determined for each medium. As the real index varies with respect to wavelength, rays are refracted at
different angles and split white light pencils into diverging pencils, commonly called *rainbows* in the case of rain drops. However, dispersion has no significant effect when the incident light is diffuse or when the medium is diffusing, because the different spectral components superpose to each other and yield again white light in all directions. This is the reason why dispersion is ignored in the case of papers or white paints and a constant real refractive index is attached to them.

Table 2. Refractive indices of materials measured at $\lambda = 589$ nm (Sodium D line)

<table>
<thead>
<tr>
<th>Material</th>
<th>Refractive Index</th>
</tr>
</thead>
<tbody>
<tr>
<td>Air$^a$</td>
<td>1.0003</td>
</tr>
<tr>
<td>Water (at 20°C)$^a$</td>
<td>1.333</td>
</tr>
<tr>
<td>Ethanol$^a$</td>
<td>1.36</td>
</tr>
<tr>
<td>Fused quartz SiO$_2$</td>
<td>1.45</td>
</tr>
<tr>
<td>Cellulose</td>
<td>1.47</td>
</tr>
<tr>
<td>Polypropylene</td>
<td>1.49</td>
</tr>
<tr>
<td>Acrylic$^b$</td>
<td>1.49</td>
</tr>
<tr>
<td>Polynvinyl alcohol</td>
<td>1.50</td>
</tr>
<tr>
<td>Plexiglass</td>
<td>1.51</td>
</tr>
<tr>
<td>Crown glass$^a$</td>
<td>1.52</td>
</tr>
<tr>
<td>Sodium Chloride (NaCl)$^a$</td>
<td>1.544</td>
</tr>
<tr>
<td>Amber$^a$</td>
<td>1.55</td>
</tr>
<tr>
<td>Polycarbonate$^b$</td>
<td>1.58</td>
</tr>
<tr>
<td>Polystyrene$^a$</td>
<td>1.59</td>
</tr>
<tr>
<td>Zircon (ZrO$_2$·SiO$_2$)$^a$</td>
<td>1.923</td>
</tr>
<tr>
<td>Diamond$^d$</td>
<td>2.417</td>
</tr>
<tr>
<td>Rutile (TiO$_2$)$^a$</td>
<td>2.907</td>
</tr>
<tr>
<td>Gold$^e$</td>
<td>0.27 + 2.95 $i$</td>
</tr>
<tr>
<td>Silver$^c$</td>
<td>0.20 + 3.44 $i$</td>
</tr>
<tr>
<td>Copper$^c$</td>
<td>0.62 + 2.57 $i$</td>
</tr>
<tr>
<td>Platinum$^c$</td>
<td>2.63 + 3.54 $i$</td>
</tr>
<tr>
<td>Aluminium$^c$</td>
<td>1.44 + 5.23 $i$</td>
</tr>
</tbody>
</table>

$^a$ Reference [6], p. 95. $^b$ Reference [35], p. 828. $^c$ Reference [2], p. 747.

### 4.2. Snell’s laws

When a light ray propagating into a given medium 1 encounters a medium 2 with different refractive index, its orientation is modified: a component is reflected back into medium 1, and a second component is refracted into medium 2. The directions of reflection and refraction satisfy Snell’s laws: 1) the incident, reflected and refracted light rays belong to a same plane, called the *incidence plane*, which also contains the normal of the interface; 2) the angles formed by the incident ray and the reflected ray with respect to the normal of the interface are equal; 3) the angle of refraction is related to the angle of incidence according to the sine law

$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$  \hspace{1cm} (30)

where $n_1$ and $n_2$ denote the refractive indices of the two media and $\theta_1$ and $\theta_2$ the respective orientations of light in them (Figure 12).
In Appendix A, we present Huygens' construction, a simple way to draw the refracted ray for a given incident angle and a given couple of refractive indices. It clearly shows that the wavelength is modified when the light enters the second medium. The wavelength concept is therefore dependent upon the medium in which light propagates, being shorter in the medium with higher index. However, as light sources and detectors are generally in air, all detected rays have their original wavelength even after having traversed different media. Wavelength variation in matter is therefore ignored and only the wavelength in air is considered.

4.3. Total reflection

Let us assume $n_1 < n_2$. When light comes from medium 1, the refraction angle is always smaller than the incidence angle. At grazing incidence, i.e. $\theta_1 = \pi/2$, the refraction angle reaches a limit value $\theta_c = \arcsin(n_1/n_2)$, called the critical angle. No light can be refracted into medium 2 with higher angle. When light comes from medium 2, it is refracted into medium 1 provided the incident angle $\theta_2$ is lower than the critical angle $\theta_c$. Otherwise, Snell’s sine law (30) provides no real solution for angle $\theta_1$, refraction does not occur and the ray is totally reflected.

4.4. Fresnel’s formulae

The fraction of light that is reflected by the interface between media 1 and 2 is called angular reflectance. It is given by Fresnel’s formulae, established by writing the transition equation of electromagnetic waves at the interface. It depends on the incident angle $\theta_1$, on the relative refractive index of the interface $n = n_2/n_1$ and on the polarization of the incident light. Most of the time, we consider unpolarized incident light which is modelled as the sum of two linearly polarized lights (see Section 1.2). Since the angular reflectance depends on the orientation of the electric field in respect to the incidence plane, we consider the cases where the electric field oscillates parallely and perpendicularly to the incidence plane. These two
polarizations are respectively called "parallel" and "perpendicular" polarizations and denoted by symbols \( p \) and \( s \).

Let us consider a light pencil coming from medium 1 with incident angle \( \theta_1 \). For \( p \)-polarized light, the angular reflectance is

\[
R_{p12}(\theta_1) = \left(\frac{\tan(\theta_1 - \theta_2)}{\tan(\theta_1 + \theta_2)}\right)^2 = \left(\frac{n \cos \theta_1 - \cos \theta_2}{n \cos \theta_1 + \cos \theta_2}\right)^2
\]

(31)

where \( \theta_2 = \arcsin(n_1 \sin \theta_1 / n_2) \) is the angle of refraction into medium 2 defined by Snell’s law. For \( s \)-polarized light, the angular reflectance is

\[
R_{s12}(\theta_1) = \left(\frac{\sin(\theta_1 - \theta_2)}{\sin(\theta_1 + \theta_2)}\right)^2 = \left(\frac{\cos \theta_1 - n \cos \theta_2}{\cos \theta_1 + n \cos \theta_2}\right)^2
\]

(32)

The variation of angular reflectance for the \( p \)- and \( s \)-polarized components are plotted in Figure 13 as a function of the incident angle \( \theta_1 \) for an interface with relative index \( n = 1.5 \).

![Figure 13](image.png)

Figure 13. Angular reflectance for \( p \)- and \( s \)-polarized lights when \( n = n_2 / n_1 = 1.5 \). The Brewster angle is \( \theta_b = \arctan(1.5) = 56.3^\circ \).

At normal incidence, \( p \)-polarized, \( s \)-polarized and unpolarized lights have the same angular reflectance:

\[
R_{p12}(0) = R_{s12}(0) = R_{12}(0) = \left(\frac{n - 1}{n + 1}\right)^2
\]

(33)

For oblique incidence, angular reflectances may expressed as functions of angle \( \theta_1 \) only, by inserting \( \theta_2 = \arcsin(\sin \theta_1 / n) \) into equations (31) and (32):

\[
R_{p12}(\theta_1) = \left(\frac{n^2 \cos \theta_1 - \sqrt{n^2 - \sin^2 \theta_1}}{n^2 \cos \theta_1 + \sqrt{n^2 - \sin^2 \theta_1}}\right)^2
\]

(34)
\[ R_{s12}(\theta_1) = \left( \frac{n^2 - \sin^2 \theta_1 - \cos \theta_1}{\sqrt{n^2 - \sin^2 \theta_1 + \cos \theta_1}} \right)^2 \] (35)

Unpolarized light contains same quantity of \( p \)- and \( s \)-polarizations. Therefore, the angular reflectance for unpolarized light is the average of the two angular reflectances:

\[ R_{12}(\theta_1) = \frac{1}{2} \left( R_{p12}(\theta_1) + R_{s12}(\theta_1) \right) \] (36)

Except at normal incidence, the \( p \)- and \( s \)-polarized lights are reflected in different proportions. The reflected and transmitted lights are therefore partially polarized. At the angle \( \theta_b = \arctan(n_{12}) \), called the Brewster angle, \( p \)-polarized light is not reflected at all. The corresponding angular reflectance is zero. The reflected light is therefore totally polarized (\( s \)-polarization). Reflection at the Brewster angle is one possible method to produce linearly polarized light.

Independently of polarization, the angular reflectance is the same if light comes from medium 1 at the angle \( \theta_1 \) or comes from medium 2 at the corresponding regular angle \( \theta_2 = \arcsin(n_1 \sin \theta_1 / n_2) \):

\[ R_{s12}(\theta_1) = R_{s21}(\theta_2) \] (37)

where symbol \( * \) means either \( s \)-polarized, \( p \)-polarized or unpolarized light.

![Graphs of reflectance and transmittance](image)

Figure 14. Angular reflectance and transmittance of an interface with relative index \( n = n_2 / n_1 = 1.5 \) as a function of the incident angle for natural light from medium 1 (left) or medium 2 (right).

Figure 14 shows the variation of angular reflectances and transmittances from normal to grazing incidence in both medium 1 and 2 for an interface with relative index \( n = 1.5 \).
Regarding the refracted component, since no light is absorbed at the interface, the *angular transmittance* is

\[ T_{s12}(\theta_1) = 1 - R_{s12}(\theta_1) \]  

(38)

and, as a consequence of (37), one has

\[ T_{s12}(\theta_1) = T_{s21}(\theta_2) \]  

(39)

This equality means that for a given path of light, the angular transmittance does not depend whether light transits from medium 1 to medium 2 or from medium 2 to medium 1. In case of total reflection, the angular transmittance is zero.

### 4.5. Radiance reflection and refraction

In radiometry, light pencils are described by the radiance concept. When a pencil enters a medium with different index, refraction modifies the ray’s geometrical extent (Figure 15). Radiance is thus modified. The relationship between incident, reflected and refracted radiances is derived from geometrical arguments issued from Snell’s laws.

![Figure 15. Incident, reflected and refracted radiances at the interface between two media of indices \( n_1 \) and \( n_2 > n_1 \).](image)

The incident radiance \( L_1 \) is defined as the flux element \( d^2\Phi_1(\theta_1, \varphi_1) \) coming from direction \((\theta_1, \varphi_1)\) through the infinitesimal solid angle \( d\omega_i = \sin \theta_1 d\theta_1 d\varphi_1 \), and illuminating an elemental area \( ds \)

\[ L_1 = \frac{d^2\Phi_1(\theta_1, \varphi_1)}{ds \cos \theta_1 \sin \theta_1 d\theta_1 d\varphi_1} \]  

(40)

The denominator in equation (40) denotes the geometrical extent of the incident pencil. Since the reflected and incident pencils form the same angle with the normal, they have the same
geometrical extent. The reflected radiance $L_R$ is therefore the incident radiance $L_1$ attenuated by the angular reflectance $R_{12}(\theta_1)$ of the interface

$$L_R = R_{12}(\theta_1)L_1 \quad (41)$$

Regarding the refracted pencil, the refraction and incidence angles satisfy Snell’s sine law (30). By differentiating equation (30), one obtains

$$n_1 \cos \theta_1 d\theta_1 = n_2 \cos \theta_2 d\theta_2 \quad (42)$$

The incident and refracted azimuthal angles form a fixed angle $\pi$, a small variation of the one implies the same variation of the other one, i.e. $d\varphi_1 = d\varphi_2$. Hence, one has

$$n_1^2 ds \cos \theta_1 \sin \theta_1 d\theta_1 d\varphi_1 = n_2^2 ds \cos \theta_2 \sin \theta_2 d\theta_2 d\varphi_2 \quad (43)$$

i.e.

$$n_1^2 dG_1 = n_2^2 dG_2 \quad (44)$$

where $dG_1$ and $dG_2$ denote the geometrical extent of the pencil in media 1 and 2 respectively. Equation (44) shows that the geometrical extent is multiplied by a factor $(n_j/n_i)^2$ each time it goes from a medium $i$ to a medium $j$, but the quantity $n_i^2 dG_i$ remains invariant. This invariance generalizes the invariance of geometrical extent stated in the previous section in the special case where the extremities of the pencil where both located in air. Finally, accounting for the changing of geometrical extent due to the refraction, the refracted radiance is

$$L_2 = (n_2/n_1)^2 T_{12}(\theta_1)L_1 \quad (45)$$

4.6. Lambertian reflectance of an interface

Let us now consider that the interface is illuminated by Lambertian light. We denote as $n = n_2/n_1$ the relative index of the interface and assume that $n > 1$. When the light comes from medium 1, the "Lambertian reflectance", denoted as $r_{12}$, is given by equation (22)

$$r_{12} = \int_{\theta_1=0}^{\pi/2} R_{12}(\theta_1) \sin 2\varphi_1 d\theta_1 \quad (46)$$

$r_{12}$ depends only on the relative index $n$. It may be computed by discrete summation with a small sampling step, e.g. $\Delta \theta_1 = 0.001$ rad. Alternatively, it is given by the following analytical formula [37], which comes from a tedious integral calculation whose main lines are presented in Appendix B.1

$$r_{12} = \frac{1}{2} + \frac{(n-1)(3n+1)}{6(n+1)^2} - \frac{2n^3(n^2+2n-1)}{(n^4-1)(n^2+1)} + \frac{8n^4(n^4+1)\ln(n)}{(n^2-1)^2(n^2+1)} + \frac{n^2(n^2-1)^2}{(n^2+1)^3} \cdot \ln\left(\frac{n-1}{n+1}\right) \quad (47)$$
The reflected flux fulfills the whole hemisphere but is not Lambertian anymore as the reflected radiance varies with angle. The transmitted flux is concentrated into the cone delimited by the critical angle \( \theta_c = \arcsin(1/n) \). The conservation of the energy at the interface implies that the transmittance is

\[ t_{12} = 1 - r_{12} \quad (48) \]

When the Lambertian light comes from medium 2, the reflectance \( r_{21} \) is similarly expressed as \( r_{12} \) with \( R_{12}(\theta_1) \) replaced by \( R_{21}(\theta_2) \)

\[ r_{21} = \int_{\theta_2=0}^{\pi/2} R_{21}(\theta_2) \sin 2\theta_2 d\theta_2 \quad (49) \]

Even though \( R_{12}(\theta_1) \) and \( R_{21}(\theta_2) \) are equal [see equation (37)], reflectances \( r_{12} \) and \( r_{21} \) are different due to total reflection which takes place in medium 2 and not in medium 1. They are related by the following formula established in Appendix B.2:

\[ 1 - r_{21} = \frac{1}{n^2} (1 - r_{12}) \quad (50) \]

One deduces from it the relationship of transmittances:

\[ t_{21} = \frac{1}{n^2} t_{12} \quad (51) \]

For an air-glass interface of typical relative index \( n = 1.5 \), one has \( r_{12} = 0.1 \), \( t_{12} = 0.9 \), \( r_{21} = 0.6 \) and \( t_{21} = 0.4 \). Their values for other indices are listed in [38] and in Appendix B.3.

### 4.7. Absorbing media and metals

The color of a homogenous medium comes from its capacity to absorb radiations of specific wavelengths in the visible domain, which corresponds to a nonzero extinction index \( \kappa(\lambda) \). The absorption coefficient \( \alpha \) is related to the extinction index by the formula [2]

\[ \alpha(\lambda) = \frac{4\pi}{\lambda} \kappa(\lambda) \quad (52) \]

This relation is valid for any absorbing medium, e.g. colored glass or metal. The particularity of metals is their high extinction index, which makes them very opaque and reflecting. The angular reflectance of air-metal interfaces is given by the same Fresnel formulas (31) and (32) as for air-dielectric interfaces, but the refractive index is a complex number \( n = n + i\kappa \) including the extinction coefficient [2]. The refraction angle \( \theta_2 \) is also a complex number. Nevertheless, the angular reflectance is real and may be expanded as follows:

\[ R_{12}(\theta_1) = \frac{\left(\sqrt{a+z} - \sqrt{2} \cos \theta_1\right)^2 + a - z}{\left(\sqrt{a+z} + \sqrt{2} \cos \theta_1\right)^2 + a - z} \quad (53) \]
\[ R_{\rho\|\|} (\theta) = R_{s\pi} (\theta) \cdot \frac{(\sqrt{a + z - \sqrt{2} \sin \theta \tan \theta})^2 + a - z}{(\sqrt{a + z + \sqrt{2} \sin \theta \tan \theta})^2 + a - z} \]  

with \( z = n^2 - \kappa^2 - \sin^2 \theta \) and \( a = \sqrt{z^2 + 4n^2\kappa^2} \). For unpolarized incident light, the angular reflectance is the average of formulas (53) and (54).

Figure 16 illustrates how the angular reflectance increases as the extinction coefficient increases. From \( \kappa = 0 \) to 0.2, the angular reflectance remains close to 0.04, i.e. the value corresponding to a real index of 1.5. This justifies that for weakly absorbing dielectrics the extinction index is not taken into account in the Fresnel formulas. Beyond 0.2, the angular reflectance increases rapidly. On a spectral point of view, the angular reflectance a surface is higher at the wavelengths where the medium is the more absorbing (higher absorption coefficient). Reflected and transmitted lights therefore get complementary colors.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure16.png}
\caption{Variation of the angular reflectance at normal incidence of an interface with relative refractive index \( \hat{n} = 1.5 + i\kappa \) as a function of \( \kappa \).}
\end{figure}

The variation of the angular reflectance as a function of the incident angle is noticeably different between absorbing media and nonabsorbing media. Figure 17 shows three examples based on the refractive indices of a strongly absorbing glass (\( \hat{n} = 1.5 + i \)), gold at 600 nm (\( \hat{n} = 0.37 + 2.82i \)) and platinum at (\( \hat{n} = 2.06 + 4.26i \)). In the three cases, the angular reflectance for \( s \)-polarized light is a strictly increasing function of the incident angle, while the one for \( p \)-polarized light decreases to a minimum without reaching zero. The reflected light is therefore partially polarized but there is no angle at which its polarization is total. In the case of gold and platinum, the angular reflectance for unpolarized light reaches a minimum, whereas the minimum is at normal incidence for dielectrics.
5. ABSORPTION

Absorption denotes the attenuation of light due to the conversion of the electromagnetic energy into another form of energy, typically because its frequencies are resonant with transition frequencies of the atoms in the medium [36]. The attenuation factor depends on the optical length traveled in the medium. It is given by the Beer-Lambert law. Absorption may be strongly dependent upon wavelength. It is responsible for the colored aspect of most objects, such as stained glasses, dyes, pigments, and inks.

5.1. Transmittance of an absorbing layer

The absorbing power of a medium is assessed by its absorption coefficient $\alpha(\lambda)$, in m$^{-1}$, which depends upon wavelength and is related to the extinction coefficient of the medium according to relation (52). According to Beer’s law, light is exponentially attenuated in this medium as a function of the path length. When a light beam crosses a layer of this medium with thickness $h$, the attenuation, called normal transmittance, is given by

$$t = e^{-\alpha h}$$

If the beam crosses the layer at an angle $\theta$ to the normal, the travelled distance in the layer becomes $h/cos\theta$ (see Figure 18) and the attenuation $T(\theta)$ becomes:

$$T(\theta) = e^{-\alpha h/cos\theta} = t^{1/cos\theta}$$ (55)
Figure 18. Path travelled through a layer with thickness $h$ by directional light oriented by an angle $\theta$ from the normal.

$T(\theta)$ is the "directional transmittance" of the layer, following the definition of "directional reflectance" given in Section 3.7. The "Lambertian transmittance" is expressed by an similar integral as in equation (22)

$$T = \int_{\theta=0}^{\pi/2} T(\theta) \sin 2\theta d\theta = \int_{\theta=0}^{\pi/2} t^{1/\cos \theta} \sin 2\theta d\theta$$ (56)

When different transparent layers with identical refractive indices are superposed, they form again an absorbing, nonscattering layer whose normal transmittance is the product of their individual normal transmittances. The oblique transmittance of the layer is also the product of the individual oblique transmittances considered at the same angle. Regarding the multilayer’s Lambertian transmittance, it is expressed by the same integral as in equation (56) where $T(\theta)$, in this case, represent the directional transmittance of the multilayer at angle $\theta$. It can be expressed in terms of the normal transmittances $t_1$, $t_2$, $t_3$, … of the different layers:

$$T = \int_{\theta=0}^{\pi/2} (t_1 t_2 t_3 \ldots)^{1/\cos \theta} \sin 2\theta d\theta$$ (57)

but not in terms of the Lambertian transmittances of the individual layers:

$$T \neq \int_{\theta=0}^{\pi/2} t_1^{1/\cos \theta} \sin 2\theta d\theta \int_{\theta=0}^{\pi/2} t_2^{1/\cos \theta} \sin 2\theta d\theta \ldots$$

5.2. Reflectance and transmittance of an absorbing film

Color filters and stained glasses are examples of absorbing layers in which light is exponentially attenuated according to Beer’s law. However, since the layer is surrounded by air, its surfaces refract and reflect light and a multiple reflection process takes place within them. Consequently, some light is reflected by the film and the global absorption is increased.

We want to determine the film's reflectance and the transmittance for natural incident light being at first collimated, then Lambertian.

Let us denote as $t$ the normal transmittance of a film considered without interfaces, and $n_2$ its refractive index. The interfaces with air ($n_1 = 1$) are assumed to be flat and parallel. The film's thickness is significantly larger than the coherence length of the incident white light, which is ordinarily a few micrometers [36]. Interferences can therefore be ignored. In the
opposite case, the film would be considered as a \textit{thin film} and its interaction with light should be described by wave optics [39].

![Figure 19: Multiple reflections of light in an absorbing plate.](attachment:figure19.png)

When the incoming pencil strikes the front surface, it splits into reflected and refracted pencils. The refracted pencil reaches the back surface where it splits again into reflected and refracted components. The reflected component again splits at the front surface into reflected and transmitted components and so on. We have a multiple reflection process as described by Figure 19. The two polarized components follow the same trajectories in the layer, which all belong to the incidence plane and form with the normal an angle $\theta$ in air and an angle $\theta'$ in the layer. The two angles are related by Snell’s sine law

$$\sin \theta = n_2 \sin \theta'$$

According to the Fresnel formulas, the surfaces’ angular reflectance and transmittance are different for the $p$ and $s$-polarized lights. Let us denote as $r_s = R_{s12} (\theta)$ the angular reflectance corresponding to reflection at angle $\theta$ in air, where the symbol * means either $p$ or $s$. Recall from equation (37) that the rays coming from the layer at the angle $\theta'$ have the same angular reflectance. The corresponding angular transmittance is $1 - r_s$.

Being given the incident angle $\theta$ of the $p$- or $s$-polarized flux $\Phi_{sj}$, we want to determine the fluxes $\Phi_{sp}$ and $\Phi_{sq}$ exiting respectively at the front side and the back side. At the front surface, $\Phi_{sj}$ splits into a reflected flux $r_s \Phi_{sj}$ and a transmitted flux $(1 - r_s) \Phi_{sj}$. The transmitted flux travels a distance $h / \cos \theta'$ in the layer and is attenuated by a factor $t^{1/\cos \theta'}$ [see equation (55)], then splits at the back surface into a reflected flux $(1 - r_s) r_{t12} t^{1/\cos \theta'} \Phi_{sj}$ and a transmitted flux $(1 - r_s)^2 t^{2/\cos \theta'} \Phi_{sj}$ which exits definitely the plate at the back side. The reflected flux is again attenuated by a factor $t^{1/\cos \theta'}$ while crossing the layer, then reaches the upper surface where it splits into a reflected component $(1 - r_s) r_{t12} t^{2/\cos \theta'} \Phi_{sj}$ and a transmitted component $(1 - r_s)^2 r_{t12} t^{2/\cos \theta'} \Phi_{sj}$ which exits the plate at the front side. By pursuing the
description of the multiple reflection process, we obtain the different fluxes exiting the plate at the upper and lower sides, the first ones being given in Figure 19. The total reflected and transmitted fluxes are expressed by the following infinite sums

$$Φ_{sI} = (1 - r_{s})^{2} t^{1/\cos θ} \sum_{k=0}^{∞} \left[ r_{s} t^{2/\cos θ} Φ_{rI} \right]$$

$$Φ_{sT} = r_{s} E_{sI} + (1 - r_{s})^{2} r_{t} t^{2/\cos θ} \sum_{k=0}^{∞} \left[ r_{t} t^{2/\cos θ} Φ_{rI} \right]$$

which are geometrical series and can be reduced. The exponents $1/\cos θ'$ can be expressed as function of angle $θ$ using the following relation issued from Snell’s law:

$$\cos θ' = \sqrt{1 - (\sin θ/n)^{2}}$$

Finally, the ratio of reflected (respectively transmitted) flux to incident flux gives the reflectance (respectively the transmittance) of the plate for the considered polarization:

$$R_{s}(θ) = R_{r12}(θ) + \frac{T_{s12}(θ) R_{s12}(θ) t^{2/\sqrt{1-(\sin θ/n)^{2}}}}{1 - R_{r12}^{2}(θ) t^{2/\sqrt{1-(\sin θ/n)^{2}}}}$$

and

$$T_{s}(θ) = \frac{T_{s12}(θ) t^{1/\sqrt{1-(\sin θ/n)^{2}}}}{1 - R_{r12}^{2}(θ) t^{2/\sqrt{1-(\sin θ/n)^{2}}}}$$

For natural light, the total reflectance and transmittance of the plate at angle $θ$ are the average of the reflectances, respectively transmittances attached to the two polarizations, i.e.

$$R(θ) = \frac{1}{2} \left[ R_{p}(θ) + R_{s}(θ) \right] \quad \text{and} \quad T(θ) = \frac{1}{2} \left[ T_{p}(θ) + T_{s}(θ) \right]$$

At normal incidence, with the angular reflectance $r_{s}$ given by equation (33), the reflectance and transmittance of the plate become

$$R(0) = \frac{8n(1-n)^{2} t^{2}}{(1+n)^{4} - (1-n)^{4} t^{2}} \quad \text{and} \quad T(0) = \frac{16n^{2} t}{(1+n)^{4} - (1-n)^{4} t^{2}}$$

The transmittance formula is especially used to assess spectral transmission by colored filters (see Reference [17], p. 30). By inverting it, we can obtain the normal transmittance $t$ from the transmittance measured at normal incidence:

$$t = \sqrt{\frac{64n^{4} + (1-n)^{4}}{(1-n)^{4} T(0) - 8n^{2}}}$$
As the incident light moves away from the normal, the s-polarized light is more reflected than the p-polarized light (see Figure 20). Consequently, if the incident light is unpolarized, the reflected and transmitted lights become partially polarized. According to the formula (1), the degree of polarization of the reflected light is

$$DOP = \frac{R_s(\theta) - R_p(\theta)}{R_s(\theta) + R_p(\theta)}$$

(66)

The variation of the DOP as a function of the incident angle is plotted in Figure 17. It is 0 at normal incidence, grows to 1 at the Brewster angle then returns to 0 at grazing incidence. This means that the reflected light remains unpolarized at normal and grazing incidences and is totally s-polarized at the Brewster angle.

In color reproduction, polarization is often ignored, which comes to consider that light is unpolarized at each reflection and refraction (see for example [60]). This yields similar reflectance and transmittance expressions as in equations (61) and (62), except that $r_s$ represents in this case the surfaces’ angular reflectance for natural light, here denoted as $r:

$$R_u(\theta) = \frac{(1-r)^2 r^2/\sqrt{1-(\sin \theta/n)^2}}{1-r^2 r^2/\sqrt{1-(\sin \theta/n)^2}} \quad \text{and} \quad T_u(\theta) = \frac{(1-r)^2 t^2/\sqrt{1-(\sin \theta/n)^2}}{1-r^2 t^2/\sqrt{1-(\sin \theta/n)^2}}$$

(67)

The variation of this reflectance $R_u(\theta)$ for a nonabsorbing plate of refractive index 1.5 is plotted in Figure 20.

![Figure 20. Directional reflectance of a nonabsorbing plate of refractive index 1.5 for the paralllely polarized light, perpendicularly polarized light and natural light according to the model accounting for polarization, and directional reflectance ignoring polarization. The Brewster angle is $\theta_b = \arctan (n_2/n_1) = 56.3^\circ$.](image)

We see that it deviates noticeably from the reflectance of an ideal plate given by equation (63) and is not valid for glass plates. However, it may happen with some kinds of plastics that
DOP is strongly decreased by loss of light’s coherence during the multiple reflection process [3], generally due to heterogeneities in the medium or the surfaces.

When the incident irradiance is Lambertian, the plate’s reflectance and transmittance are given by similar formulas as for a single interface [see equation (46)], i.e.

\[
\tilde{r} = \int_{\theta=0}^{\pi/2} R(\theta) \sin 2\theta d\theta \quad \text{and} \quad \tilde{t} = \int_{\theta=0}^{\pi/2} T(\theta) \sin 2\theta d\theta
\]

(68)

where \( R(\theta) \) and \( T(\theta) \) are given in equation (63) or, if polarization is ignored, in equation (67). In the case of the nonabsorbing plate of refractive index of 1.5, the difference between the models with and without account for polarization is 4% for the Lambertian reflectance, and less than 1% for the Lambertian transmittance. This can justify that polarization is ignored for transmission filters in applications where no much accuracy is needed, for example in the color assessment of stained glasses or filtered lightings.

6. SURFACE SCATTERING

Compared to a flat interface, a rough interface reflects and transmits collimated incident light into an enlarged set of directions. The topography of the rough interface has a random elevation as featured in Figure 21. The elevation function is modeled by a probability distribution parameterized by a characteristic vertical length, the root-mean-square (r.m.s.) height \( \sigma \), and by a characteristic horizontal length, the correlation length \( \tau \) [9]. Another parameter is also commonly used: the r.m.s. slope \( m \), corresponding to the ratio \( \sigma/\tau \) [40].

![Figure 21. Profile of the elevation function \( e(\mathbf{x},\mathbf{y}) \) of a rough interface along the \( x \)-axis. The random pattern has a r.m.s. height \( \sigma \) and a correlation length \( \tau \). The r.m.s. slope \( m \) of the interface is the ratio \( \sigma/\tau \).](image)

Most models assume that the local slope within rough interfaces follows a Gaussian distribution [41, 9]. In order to ease the application of optical laws, local slope is converted into local normal vector [42] denoted by the differential solid angle \( d\omega_h = \sin \theta_h d\theta_h d\phi_h \) (Figure 22). For an isotropic Gaussian distribution of slopes, the probability distribution function \( D \) of the normal vector orientations is
This function is known as the *Beckmann function* [9,43,44]. It depends only on the polar angle $\theta_h$ due to the assumption of roughness azimuthal isotropy.

Figure 22. 2D representation of a rough interface. The directional incident light (direction denoted by the differential solid angle $d\omega_i$) hits a small portion of interface having the normal vector $d\omega_h$. It is reflected and transmitted into directions $d\omega_r$ and $d\omega_t$ respectively.

### 6.1. BRDF and BTDF models

The reflectance and transmittance of rough interfaces can be deduced from their BRDF, respectively their BTDF using equation (21). BRDFs and BTDFs may be determined experimentally [45,46,9] or computed thanks to an optical model. The model is derived from equations relying on either physical or geometrical optics depending on the size of the roughness patterns [44].

Physical optics models are directly based on the electromagnetic wave theory and Maxwell’s equations [2]. They shall be used when the wavelength of light is large or comparable to the r.m.s. height $\sigma$ and the correlation length $\tau$. In such a case, the diffraction of the incident waves by the corrugations of the interface is dominant. It is assumed that the interface does not have any discontinuity or sharp arc compared to the wavelength of incident light. It may therefore be represented locally by its tangent plane, on which light is reflected according to Snell’s law and diffracted because of the small size of the plane. This *tangent plane approximation* is the basis of Beckmann’s model [41], also known as *Kirchhoff’s approximation* [47].

Models relying on geometrical optics models explain the behavior of light when its wavelength is small compared to the roughness characteristic lengths. Diffraction becomes
negligible. **Slope distribution models**, such as the well-known models developed by Torrance and Sparrow [40] and by Cook and Torrance [43] consider the rough interface as a set of randomly inclined microfacets reflecting and transmitting light like flat interfaces. According to slope distributions models [40, 43, 44], the BRDF $f_R$ of a rough interface is

$$f_R(d\omega_i,d\omega_r) = \frac{D(\theta_h)G(\theta_i,\theta_r)R_{12}(\theta'_r)}{4\cos\theta_i\cos\theta_r} \tag{70}$$

When the medium of transmission is non-metallic, the BTDF is [48]

$$f_T(d\omega_i,d\omega_t) = \frac{D(\theta_h)G(\theta_i,\theta_r)T_{12}(\theta'_r)}{\Gamma(\theta'_r)\cos\theta_i\cos\theta_r} \tag{71}$$

In Eqs. (70) and (71), differential solid angles $d\omega_i = \sin\theta_i d\theta_i d\phi_i$, $d\omega_r = \sin\theta_r d\theta_r d\phi_r$ and $d\omega_t = \sin\theta_t d\theta_t d\phi_t$ denote respectively the directions of incidence, of reflection and of transmission (see Figure 22), angle $\theta_h$ represents the inclination of the interface’s local normal vector, which is related to the angles of incidence and reflection by

$$\theta_h = \arccos\left[\frac{\cos\theta_i + \cos\theta_r}{\sqrt{2\left(1 + \cos\theta_i\cos\theta_r + \sin\theta_i\sin\theta_r\cos(\phi_r - \phi_i)\right)}}\right], \tag{72}$$

angle $\theta'_r$ denotes the local angle of incidence of light, which is related to the angles of incidence and reflection by

$$\theta'_r = \frac{1}{2}\arccos\left(\cos\theta_i\cos\theta_r - \sin\theta_i\sin\theta_r\cos(\phi_r - \phi_i)\right), \tag{73}$$

function $D$ is the probability distribution function of the local normal vector, given by equation (69), function $G$ is a shadowing function that is presented below, $R_{12}$ is the Fresnel angular reflectance of the interface and $\Gamma(\theta'_r)$ expresses the spreading of the transmitted solid angle due to the refraction by the interface:

$$\Gamma(\theta'_r) = \left(\cos\theta'_r + \sqrt{n^2 - 1 + \cos\theta'_r}\right)^2 \tag{74}$$

A rough interface may comprise shadow areas, which increase with the roughness and the incidence angle of light. Interface elements belonging to shadow areas do not contribute to the reflection nor the transmission. This phenomenon, illustrated by Figure 23, is called shadowing.

Likewise, reflected and transmitted light may be partially blocked by neighboring corrugations. This phenomenon, sometimes called masking [40], is equivalent to shadowing but depends on the angle of observation instead of the angle of incidence.
The fraction of facets that really contribute to the reflection of light from direction $d\omega_i$ to direction $d\omega_r$ is given by function $G(\theta_i, \theta_r)$, product of two similar functions $g$, one for shadowing, and the other one for masking

$$G(\theta_i, \theta_r) = g(\theta_i) g(\theta_r)$$  \hspace{1cm} (75)

Using a statistical model, Smith computed the following shadowing function $g$ [49]

$$g(\theta, \theta') = \begin{cases} 
\frac{1}{\Lambda_m(\theta)} + 1 & \text{if } \cos\theta' > 0 \\
0 & \text{if } \cos\theta' < 0 
\end{cases}$$  \hspace{1cm} (76)

where $\theta'$ is the local angle of incidence given by (73) and $\Lambda_m$ is a function of angle $\theta$ which depends on the r.m.s. slope $m$:

$$\Lambda_m(\theta) = \frac{1}{2} \left[ \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{2m}}{\cot \theta} \exp\left(\frac{-\cot^2 \theta}{2m^2}\right) - \text{erfc}\left(\frac{\cot \theta}{\sqrt{2m}}\right) \right]$$

Function $g$ is comprised between 0 (facets completely shadowed or masked) and 1 (facets completely illuminated). At small and medium incidence angles, the illuminated fraction of the facet’s area is close to 1. The shadowing effect is thus small enough to be neglected. However, ignoring the shadowing at high incidence angles may yield an overestimation of the reflected and transmitted fluxes, and a subsequent violation of the energy conservation principle. According to Bruce [50] and Caron [51], shadowing should be taken into account when the incidence angle is higher than a limit angle $\theta_{shad}$ depending on the r.m.s. slope $m$ of the rough interface

$$\theta_{shad} = \frac{\pi}{2} - \arctan\left(\sqrt{2m}\right)$$  \hspace{1cm} (77)

The same considerations apply for masking.

6.2. Gloss

The light component reflected at the surface of the objects is generally well distinguishable from the component having entered the objects’ material. It gives rise to a different perceptual
attribute called *glossiness*. This surface reflection component is not or little colored in comparison to the light issued from the matter which is subject to wavelength-dependent absorption. Moreover, the angular distribution of the two components may be very different, especially when the object is diffusing and its surface is smooth or polished.

The study of gloss perception is more recent than the study of color [52] and there is no normalized gloss perception space available today. The main approach consists in correlating gloss perception, surface topology and BRDF measurements [53], but in the case of colored objects, it is still difficult to assess color and gloss attributes from optical measurement [25]. First attempts of gloss assessment are due to Hunter, Judd and Wyszecki [54] but according to Wills, Agarwal, Kriegman and Belongie [55], the modern notion of gloss was formalized by the American Society for Testing and Materials (ASTM) as the angular selectivity of reflectance, involving surface-reflected light, responsible for the degree to which reflected highlights or images of objects may be seen as superimposed on a surface [56]. In order to cope with the variety of materials and gloss effects, several types of glossiness are defined, each one being assess by measurement with a specific $\theta_i$: $\theta_r$: bidirectional geometry [16]:

- **specular gloss** is the perceived brightness associated with the specular reflection from a surface (measurement geometries: 20°:20°, 45°:45° and 60°:60°),
- **sheen** is the perceived shininess from matte surfaces at grazing angles (85°:85°),
- **Distinctness of image (DOI)** is the perceived sharpness of images reflected in a surface (30°:30.3°),
- **Bloom**, also called 2° Haze, is the perceived cloudiness in reflections near the specular direction (30°:32°),
- **Haze** is the shininess measured at 5° to the specular direction (30°:35°),
- **diffuseness** is the perceived brightness for diffusely reflecting areas (30°:45°) and
- **contrast gloss** is the perceived relative brightness of specularly and diffusely reflecting areas (45°:45° and 45°:0° geometries).

## 7. VOLUME SCATTERING

As light encounters small fluctuations of refractive index within the medium, a portion of the incident light is scattered. In the atmosphere, scattering yields the white color of clouds (Mie scattering [57]), the blue color of the sky and the redness of sunsets (Rayleigh scattering [58]). Scattering also occurs in liquids. Milk, for example, is composed of a suspension of almost transparent fat droplets which scatter light and give milk its white and opaque aspect. In the case of oceans, scattering is coupled to absorption, which produces the characteristic bluish color. Light is also scattered in solid heterogeneous media, such as paintings, papers, cotton and human tissues [59]. Different types of scattering are encountered according to the composition, shape, size and concentration of the heterogeneities, often considered as particles immerged into a binder. The polarization and the wavelength of the incident light
may have a strong influence on scattering. We present here some commonly used parameters and models relative to scattering.

7.1. Scattering description parameters

A collimated beam traversing a path of length $x$ into a scattering and absorbing medium undergoes an exponential attenuation $T$ described by the Beer-Lambert law

$$T = e^{-K_{\text{ext}}x}$$

where $K_{\text{ext}}$ is the linear extinction coefficient (in $\text{m}^{-1}$). The inverse of the extinction coefficient is the extinction free-mean-path length $l_{\text{ext}}$, characterizing the distance along which directional flux is attenuated by a factor $1/e$

$$l_{\text{ext}} = 1/K_{\text{ext}}$$

The linear extinction coefficient may be decomposed into a component $K_{\text{sca}}$ related to scattering and a component $K_{\text{abs}}$ related to absorption

$$K_{\text{ext}} = K_{\text{sca}} + K_{\text{abs}}$$

Free-mean-path lengths are also defined for scattering $l_{\text{sca}}$ and for absorption $l_{\text{abs}}$

$$l_{\text{sca}} = 1/K_{\text{sca}} \quad \text{and} \quad l_{\text{abs}} = 1/K_{\text{abs}}$$

The scattering and absorbing medium is said to be homogenous when its coefficients $K_{\text{sca}}$ and $K_{\text{abs}}$ are independent of position. These coefficients are generally functions of wavelength. Beer’s law corresponds to the special case where $K_{\text{sca}} = 0$.

As an effect of scattering, the trajectory of light is modified. The change of direction in an elementary volume of medium is specified by a volume angular scattering coefficient (VSF) [8], defined for every direction $(\theta, \phi)$ as

$$\beta_\lambda(\theta, \phi) = \frac{d^2\Phi_\lambda(\theta, \phi)}{E_\lambda d\omega dV} = \frac{dl_\lambda(\theta, \phi)}{E_\lambda dV}$$

where $d^2\Phi_\lambda$ denotes the element of spectral flux scattered out of the volume $dV$ into the elemental solid angle $d\omega$, $E_\lambda$ the incident collimated spectral irradiance and $dl_\lambda = d^2\Phi_\lambda / d\omega$ the scattered element of spectral intensity. The VSF integrated over the $4\pi$ sr solid angle gives the linear scattering coefficient $K_{\text{sca}}(\lambda)$. Hence, dividing the VSF $\beta_\lambda(\theta, \phi)$ by $K_{\text{sca}}(\lambda)$ yields a normalized function called angular scattering distribution function $f_S(\lambda, \theta, \phi)$ satisfying the following normalization condition for each wavelength

$$\int_{(\theta, \phi) \in 4\pi} f_S(\lambda, \theta, \phi) d\omega = 1$$

Even though $f_S$ is independent of absorption, it remains a function of wavelength as directions of scattering generally depend upon wavelength especially when scattering is due
to diffraction (Rayleigh scattering, Mie scattering [2]). The rigorous definition for \( f_S \) is the ratio of scattered element of intensity \( dI_\lambda = d^2\Phi_\lambda / d\omega \) to total scattered element of flux \( d\Phi_\lambda \)

\[
f_S(\lambda, \theta, \varphi) = \frac{dI_\lambda(\theta, \varphi)}{d\Phi_\lambda}
\]  

(84)

Equation (83) comes from the fact that the total scattered spectral flux \( d\Phi_\lambda \) is the sum of all spectral intensities over the \( 4\pi \) sr solid angle. If scattering by the volume element is isotropic, equal intensity is emitted in every direction and function \( f_S \) is a constant equal to \( 1/4\pi \). The ratio of the function \( f_S \) of a given medium to the one of an isotropic diffuser is called the phase function, denoted as \( P \):

\[
P(\lambda, \theta, \varphi) = 4\pi f_S(\lambda, \theta, \varphi)
\]  

(85)

Figure 24 shows an example of phase function represented in one plane containing the incident beam. As a consequence of equations (83) and (85), the normalization equation for the phase function is

\[
\frac{1}{4\pi} \int_{(\theta, \varphi) \in 4\pi} P(\theta, \varphi) \, d\omega = 1
\]  

(86)

In the case of isotropic scattering, the phase function is 1 in all directions. In the opposite case, anisotropic scattering may be characterized by an anisotropy parameter \( g \) defined as the average cosine of the scattering angle

\[
g = \frac{1}{4\pi} \int_{(\theta, \varphi) \in 4\pi} P(\theta, \varphi) \cos \theta \, d\omega = 1
\]  

(87)

The incident light is mainly scattered backwards when \( g \) is close to \(-1\) or forwards when \( g \) is close to \(1\). For isotropic scatterings, \( g = 0 \). Parameter \( g \) is used for defining the transport free-mean-path length \( l_{\text{trans}} \), corresponding to the distance from which one may consider that light has completely lost the memory of its original direction of incidence

\[
l_{\text{trans}} = \frac{l_{\text{scat}}}{1 - |g|}
\]  

(88)
The optical thickness $\tau$ of a scattering or/and absorbing layer having a thickness $h$ and an extinction coefficient $K_{\text{ext}}$ is defined as

$$\tau = K_{\text{ext}} h$$  \hspace{1cm} (89)

When $\tau \gg 1$, a directional incident light is almost completely attenuated. When $\tau$ is small, the layer is translucid, i.e. we can distinguish an object located beneath the layer. After a certain number of scattering events, light propagates in an isotropic manner, i.e. it becomes Lambertian.

### 7.2. Types of scattering

The notion of optical thickness defined above allows estimating the number of scattering events that a light ray undergoes across a given layer of the considered medium. In the particular case of a weakly absorbing medium ($K_{\text{abs}} \ll K_{\text{sca}}$), the optical thickness describes the strength of scattering. We may distinguish four scattering modes, according to the value of the optical thickness of the layer:

- **ballistic scattering**, also called atmospheric absorption [7], in which light is almost not scattered: $\tau \ll 1$ and $h \ll l_{\text{sca}}$.

- **single scattering** in which light is scattered once in the medium: $\tau \approx 1$ and $h \approx l_{\text{sca}}$. For particle sizes much smaller than the wavelength such as air molecules, smoke and dust, Rayleigh scattering [60, 58, 61] is applicable with the following phase function for unpolarized light:

  $$P_R(\theta, \phi) = \frac{3}{8} \left(1 + \cos^2 \theta\right)$$  \hspace{1cm} (90)

For larger particles with size comparable to the wavelength, Mie scattering [57, 16, 2] becomes applicable and is often represented by approximated phase functions such as the famous Henyey-Greenstein phase function [62] parameterized by the anisotropy parameter $g$ defined in equation (87)

$$P_{\text{HG}}(\theta, \phi) = \frac{1 - g^2}{\left(1 - 2g \cos \theta + g^2\right)^{3/2}}$$  \hspace{1cm} (91)

When particle sizes are much larger than the wavelength, geometrical optics models may be used [63, 64].

- **multiple scattering** in which light is scattered various times [65]: $\tau > 1$ and $h > l_{\text{sca}}$.

- **diffusion** where scattering events occur so many times that the resulting scattering is isotropic: $\tau \gg 1$ and $h \gg l_{\text{sca}}$. According to Eq. (89), since parameter $g$ defined by (87) is equal to 0,
the transport length is given by the scattering length. The incident light has therefore completely lost the memory of its incident direction.

For low concentrations of particles, it is assumed that they do not interact one with each other. Scattering is said to be independent. Describing the scattering by one particle is sufficient to determine the scattering by the whole medium. For high concentrations of particles, scattering becomes dependent. In the case of independence, geometrical optics may be used when the size of the particles are large compared to the wavelength of the incident light. However, when the particles are small compared to the wavelength, light is diffracted. In this case, scattering may be modeled by the Rayleigh scattering theory. The Mie scattering theory describes the diffraction of light by spherical particles of complex refractive index in a dielectric medium (real refractive index). Note that, excepted for exceptional phenomena such as the Raman effect, scattering does not modify the wavelength of the incident light and is thus said to be elastic [36].

7.3. The radiative transfer equation

In many applications, a simple phenomenological approach, based on the notion of directed light ray and conservation of energy, provides a realistic description of the scattering phenomenon. Considering a sufficiently large portion of the heterogeneous medium, the scattering process is described by a simple equation: the radiative transfer equation [66]. A priori, it is valid only when the scattering free-mean-path length \( l_{sca} \) is large compared to the wavelength of the incident light and to the dimension of the heterogeneities responsible for the scattering, but specific studies have shown that its domain of validity can be enlarged to other cases.

The radiative transfer equation expresses the conservation of the radiant flux in a given element of volume and a given direction. This energy balance shall be performed everywhere in the medium and in every direction. Let us consider a small cylinder of section \( dS \) and of length \( dl \) oriented according to the incident direction \( u \). Radiance \( L(u) \) decreases along this direction due to absorption and scattering

\[
\frac{dL(u)}{dl} = -(K_{abs} + K_{sca}) L(u)
\]

(92)

At the same time, the cylinder receives radiances \( L(u') \) from all directions \( u' \) and scatters them partially towards direction \( u \). The portion of radiance \( L(u') \) that contributes to radiance \( L(u) \) is \( \frac{K_{sca}}{4\pi} P(u', u) L(u') d\omega' \), where \( P(u', u) \) is the phase function of the considered cylindrical element of volume. By summing up the contributions of all directions \( u' \) and adding the resulting global contribution to equation (92), one obtains the radiative transfer equation.
\[
\frac{dL(u)}{dl} = \left( -K_{\text{abs}} + K_{\text{sca}} \right) L(u) + \frac{K_{\text{sca}}}{4\pi} \int_{\Sigma} P(u', u) L(u') \, d\omega' \quad (93)
\]

This equation has no general solution. An exact or approximated solution must be searched for every particular scattering medium. Various solutions have been developed. Let us mention the main ones:

The \textit{N}-fluxes method \cite{67}, which allows converting the integrodifferential equation (93) into a differential equation system thanks to an angular discretization. Solutions are obtained for azimuthally isotropic media, the discretization being performed only according to the zenithal angle \cite{68}; the simplest particular case is the two-flux Kubelka-Munk model \cite{69, 70}. Four-flux models have also been developed \cite{71, 72}.

The discrete ordinate method \cite{73}, which is an exact but computationally expensive method. The assumption of azimuthal isotropy is not necessary. Discretization according to the azimuthal angle is avoided thanks to a Fourier series development for the scattered fluxes and a spherical harmonic decomposition of the phase function.

The auxiliary function method \cite{74}, which avoids angular discretization. An auxiliary function is introduced into the radiative transfer equation and decomposed into spherical harmonics. The radiative transfer equation is thus converted into an integral equation system, which can be solved numerically.

The adding-doubling method \cite{75, 59} is based on an infinitesimal sublayer whose optical properties are described by a matrix. The \((i, j)\)-entries of the matrix, deduced from the phase function, indicate the probability for a ray coming from direction \(i\) to be scattered into direction \(j\). Then, the matrix is squared, or raised to a power \(k\), to represent the optical properties of a sublayer with double thickness, respectively with thickness multiplied by \(k\). This doubling or adding operation is repeated until the thickness of the sublayer match the whole considered layer thickness.

\subsection*{7.4. Scattering in Lambertian layers}

The free-mean-path length \(l_{\text{sca}}\) of a strongly scattering layer is very small compared to the layer thickness. Incident light looses the memory of its initial angular distribution as soon as it penetrates the layer. We can therefore assume that any illumination geometry leads to a same reflectance and a same transmittance, called \textit{intrinsic}. Since light is scattered a large number of times within the layer, it is Lambertian at every point of the layer, especially at the layer’s bounding planes. We can also assume that light exiting the layer is unpolarized.

The reflection and transmission by Lambertian layers can be modeled by the Kubelka-Munk two flux model \cite{69, 70}, with a satisfying accuracy if the layer is weakly absorbing \cite{68, 76}. 
8. THE KUBELKA-MUNK MODEL

The scattering model proposed by Kubelka and Munk [69, 70] was initially introduced for predicting the reflectance of paints, but it has been also used in a wide range of domains where uniform and infinitely large layers of a scattering medium are encountered. It corresponds to a special case of the radiative transfer theory where the phase function is reduced to a pair of opposite directions or, more precisely, of opposite sets of directions covering respectively the upper and the lower hemispheres [77]. It is thus often qualified as a two-flux model. The interest of the Kubelka-Munk model lies in the simple differential equation system expressing the scattering and absorption phenomena within the layer. The differential equations involve the upward and the downward oriented fluxes, which are functions of depth in the layer. Another interest of this model is the fact that the solutions of the differential equation system have analytical expression.

8.1. Kubelka-Munk differential equation system

The Kubelka-Munk model considers a thin slice of material whose thickness \( dz \) is small compared to its two other dimensions (see Fig. 25). Let us denote as \( j \) a diffuse flux oriented upwards and as \( i \) a diffuse flux oriented downwards. We assume that all variations of fluxes are due to absorption and scattering. Let us denote \( K \) the linear absorption coefficient and \( S \) the linear scattering coefficient.

![Fig. 25. Upward and downward diffuse fluxes crossing a sublayer of thickness \( dz \).](image)

By crossing the slice of matter, the flux \( i(z) \) decreases by an amount \((Kdz)i(z)\) due to absorption and also by an amount \((Sdz)i(z)\) due to scattering. However, it increases by the amount of flux \( j(z) \) lost by scattering while crossing the same slice of material in the opposite direction: \((Sdz)j(z)\). Therefore we get:

\[
\frac{di}{dz} = -(K + S)i + Sj
\]

(94)

Analyzing the flux \( j \) leads to a similar equation. The orientation of this flux being opposite, we need to introduce negative signs:
\[
\frac{dj}{dz} = (K + S)j - Si \tag{95}
\]

Combining equations (94) and (95) gives a system of differential equations:

\[
\begin{align*}
\frac{di}{dz} &= -(K + S)i + Sj \\
\frac{dj}{dz} &= -Si + (K + S)j
\end{align*} \tag{96}
\]

### 8.2. Solving the differential equation system

There are several ways to solve equation (96), we will present two approaches. The first one uses the Laplace transform and the second one uses the matrix exponential. The Laplace transform \([78]\) associates to a causal function \(f(x)\) the function \(F(p)\) such that

\[
F(p) = \int_0^\infty f(t) e^{-pt} \, dt \tag{97}
\]

It is a linear transform. The derivative of a function \(f\) is transformed into \(pf(p) - f(0)\), where the constant \(f(0)\) is the value of \(f\) at \(x = 0\). Considering the Laplace transform of Eq. (96), we get the following system:

\[
\begin{align*}
pl(p) - i(0) &= -(K + S)I(p) + SJ(p) \\
pJ(p) - j(0) &= SI(p) - (K + S)J(p)
\end{align*} \tag{98}
\]

where \(I(p)\) and \(J(p)\) are the Laplace transforms of \(i(z)\) and \(j(z)\) respectively. Solving Eq. (98) yields:

\[
\begin{align*}
I(p) &= \frac{(p - aS)i(0) + Sj(0)}{p^2 - b^2S^2} \\
J(p) &= \frac{(p + aS)j(0) - Si(0)}{p^2 - b^2S^2}
\end{align*} \tag{99}
\]

with

\[
a = (K + S)/S \quad \text{and} \quad b = \sqrt{a^2 - 1} \tag{100}
\]

When a function has a Laplace transform, this latter is unique. The converse is also true. Since \(p/(p^2 - b^2S^2)\) and \(bS/(p^2 - b^2S^2)\) are the respective Laplace transforms of \(\cosh(bSz)\) and \(\sinh(bSz)\) \([79]\), one concludes that \(I(p)\) and \(J(p)\) are the respective Laplace transforms of

\[
i(z) = i(0)cosh(bSz) + \frac{1}{b}[j(0) - ai(0)]\sinh(bSz) \tag{101}
\]

and
Expressions (102) and (101) are the general solutions of the Kubelka-Munk differential equation system.

The second approach looks at equation (96) as a differential equation in a vector space that can be written as

\[
\begin{pmatrix}
\frac{d}{dz} j(z)
\end{pmatrix} = \begin{pmatrix}
-(K+S) & S \\
-S & (K+S)
\end{pmatrix} \begin{pmatrix}
j(z) \\
i(z)
\end{pmatrix}
\tag{103}
\]

The solution of this Eq. is given by the matrix exponential of the matrix [80]:

\[
\begin{pmatrix}
j(z) \\
i(z)
\end{pmatrix} = \exp \left[ \begin{pmatrix}
-(K+S) & S \\
-S & (K+S)
\end{pmatrix} (z-0) \right] \begin{pmatrix}
j(0) \\
i(0)
\end{pmatrix}
\tag{104}
\]

where the matrix exponential is defined by similar series as the classical exponential:

\[
\exp(M) = \sum_{k=0}^{\infty} \frac{M^k}{k!} = 1 + M + \frac{M^2}{2} + \frac{M^3}{6} + \frac{M^4}{24} + \ldots
\tag{105}
\]

Let us now derive the reflectance and transmittance of a layer of thickness \( h \) considered without interface. Denoting as \( I_0 \) incident flux on the front side, the layer's reflectance and transmittance correspond to the flux ratios \( j(0)/I_0 \), respectively \( i(h)/I_0 \). As boundary conditions, we have:

\[
i(0) = I_0
\tag{106}
\]

and

\[
j(h) = 0
\tag{107}
\]

By setting these boundary conditions in equations (101) and (102), or equivalently into Eq. (104), at depth \( z = h \), we obtain the following reflectance and transmittance formulas

\[
\rho_h = \frac{j(0)}{I_0} = \frac{\sinh(bSh)}{b \cosh(bSh) + a \sinh(bSh)}
\tag{108}
\]

and

\[
\tau_h = \frac{i(h)}{I_0} = \frac{b}{b \cosh(bSh) + a \sinh(bSh)}
\tag{109}
\]

with \( a \) and \( b \) given in Eq. (100).
8.3. Infinitely thick layer

Let us consider the particular case of a layer whose thickness becomes infinite. From Eq. (109), we can compute the limit as shown in Eq. (110)

\[
\lim_{h \to \infty} \rho_h = \lim_{h \to \infty} \frac{1}{a + b \coth(bSh)} = \frac{1}{a + b} = a - b
\]  

(110)

Let us denote this limit as \( \rho_\infty \). In literature (see Ref. [17], p.785), \( \rho_\infty \) is expressed as a function of \( K \) and \( S \)

\[
\rho_\infty = 1 + \frac{K}{S} - \sqrt{\left(\frac{K}{S}\right)^2 + 2\frac{K}{S}}
\]  

(111)

The ratio between \( K \) and \( S \) can be expressed as a function of \( \rho_\infty \)

\[
\frac{K}{S} = \frac{(1-\rho_\infty)^2}{2\rho_\infty}
\]  

(112)

Eq. (112) is the most popular result from the Kubelka-Munk theory.

An observer perceives a layer of finite thickness \( h \) as infinitely thick if no light emerges on the other side. In practical terms this is an opaque layer and yields the boundary condition \( i(h) = 0 \). Combined with the boundary conditions expressed in equations (106) and (107), Eq. (101) becomes

\[
I_0 \cosh(bSh) + \frac{j(0) - aI_0 \sinh(bSh)}{b} = 0
\]  

(113)

thus providing

\[
\rho_\infty = a \sinh(bSh) - b \cosh(bSh)
\]

8.4. Layer in optical contact with a background

Many practical cases, like ink on paper or paint on a substrate, can be seen as layers in optical contact with a background having a reflectance factor \( \rho_g \). Now the boundary condition at \( x = h \) are given by

\[
j(h) = \rho_g j(h)
\]  

(114)

Substituting equations (114) and (106) in equations (101) and (102) leads to:

\[
\rho = \frac{(1-\rho_g) \sinh(bSh) + b\rho_g \cosh(bSh)}{(a-\rho_g) \sinh(bSh) + b \cosh(bSh)}
\]  

(115)

Assuming \( bSh \neq 0 \), Eq. (115) can be simplified as:
\[
\rho = \frac{1 - \rho_g (a - b \coth(bsh))}{a - \rho_g + b \coth(bSh)}
\]  
(116)

Equation (116) is called the hyperbolic solution of the Kubelka-Munk equations. As \(bSh\) tends to 0, \(\rho\) tends towards \(\rho_g\).

Two particular cases of interest are the reflectances \(\rho_0\) and \(\rho_1\) of a layer in optical contact with an ideal black background and an ideal white background, respectively:

\[
\rho_0 = \frac{\sinh(bSh)}{a \sinh(bSh) + b \cosh(bSh)}
\]  
(117)

and

\[
\rho_1 = \frac{(1-a) \sinh(bSh) + b \cosh(bSh)}{(a-1) \sinh(bSh) + b \cosh(bSh)}
\]  
(118)

**8.5. Saunderson correction**

The reflectances modeled by the previous equations do not take into account the reflections of light at the interface with air, whereas these reflections are significant and cannot be ignored. Saunderson proposed a correction for the Kubelka-Munk reflectance formula taking them into account [81].

Let us denote as \(R\) the effective reflectance of the layer and \(\rho\) the reflectance it would have without interface given by the Kubelka-Munk model. Reflectance \(\rho\) is defined as the ratio of upward to downward fluxes at depth 0, i.e. \(\rho = j(0) / i(0)\). The effective reflectance is the ratio of outgoing to incoming fluxes, i.e. \(R = J_0 / I_0\). Regarding the flux coming from air, a fraction \(r_e\) of it (external reflectance) is reflected and a fraction \(T_{in}\) is transmitted into the medium (see Fig. 26). Regarding the upward flux \(j(0)\) coming from the medium, a fraction \(r_i\) of it (internal reflectance) is reflected and a fraction \(T_{out}\) is transmitted to air.

![Fig. 26. Reflections and transmissions of diffuse light at the air-medium interface.](image)

We have:
\[ i(0) = T_{in}I_0 + r_i j(0) \]
\[ J_0 = r_e I_0 + T_{out} \rho i(0) \]  

(119)

From these two equations, we derive the Saunderson correction formula:
\[ R = r_e + \frac{T_{in}T_{out} \rho}{1 - r \rho} \]  

(120)

The terms \( r_e, r_i, T_{in}, \) and \( T_{out} \) depend only on the refractive index of the medium. Assuming that the interface is flat, they are derived from the Fresnel formula in respect to the illumination and observation geometries.

In the case of a diffuse-diffuse geometry, the incident flux is Lambertian and the whole reflected light is captured by the detector. We thus have: \( r_e = r_{12} \) given by Eq. (47), \( r_i = r_{21} \) deduced from \( r_{12} \) according to Eq. (50), \( T_{in} = t_{12} = 1 - r_{12} \) and \( T_{out} = t_{21} = 1 - r_{21} \). In the case of diffuse-directional geometry, where the incident light is Lambertian and only the radiance in one direction (at angle \( \theta \) to the normal) is observed, we rather have: \( r_e = R_{01}(\theta) / \pi \) and \( T_{out} = T_{01}(\theta) / (n^2 \pi) \), where \( R_{01}(\theta) \) and \( T_{01}(\theta) \) denote the Fresnel angular reflectance, respectively angular transmittance, of the interface. The factors \( 1 / \pi \) come from the fact that the radiance exiting towards the observer forms only a fraction \( 1 / \pi \) of the total incoming fluxes (see Section 3.5), but they remove if one considers the reflectance factor in respect to a perfectly white diffuser. The term \( 1 / n^2 \) in \( T_{out} \) comes from the changing of solid angle due to the refraction of exiting rays through the interface, as explained in Section 4.5.

### 8.6. Saunderson correction in transmittance mode

When a layer of diffusing medium is observed by transmission of light, the reflections and transmissions of light at the interfaces may be modeled in a similar manner as in reflectance mode, except that the incident flux \( I_0 \) illuminates the back air-medium interface. We denote as \( \rho \) and \( \tau \) the intrinsic reflectance, respectively intrinsic transmittance of the layer, and as \( r_i \) and \( r_i' \) the internal reflectance of the top interface, respectively the back interface for Lambertian light. The flux transfers represented in Fig. 26 enable writing the following equations:

\[ i(0) = r_i j(0) \]
\[ j(0) = \rho i(0) + \tau j(h) \]
\[ i(h) = \rho j(h) + \tau i(0) \]
\[ j(h) = T_{in} I_0 + r_i' j(h) \]
\[ J_0 = T_{out} j(0) \]  

(121)

We deduce the expression for the layer's transmittance with interfaces:
\[
T = \frac{J_0}{I_0} = \frac{T_{in}T_{out}\tau}{(1-r_\rho)(1-r'_\rho) - \eta_\rho\tau^2}
\] (122)

Since the layer is bounded by air at its two sides, the two internal interface reflectances are: \(r_i = r'_i = r_{21}\). The terms \(T_{in}\) and \(T_{out}\) are similar as those presented in the previous section for the reflectance mode.

Fig. 27. Reflections and transmissions of diffuse light at the air-medium interfaces in transmittance mode.

8.7. Validity of the Kubelka-Munk model

The Kubelka-Munk model became famous in many technical domains where light scattering is concerned, mainly owing to its reduced number of parameters (absorption coefficient, scattering coefficient and medium thickness) and a set of simple formulas relating these parameters with measurement easily performable in practice. However, in many cases, large deviation between prediction and measurement is observed due to the fact that the medium optical properties and/or the light and measurement conditions are not compatible with the assumptions underlying the model. Abundant literature has been dedicated to the statement of the Kubelka-Munk limitations as well as improvements for specific applications. As a special case of the radiative transfer theory, the Kubelka-Munk model relies on the following assumptions: the average distance between diffuser and/or absorbers should be very large compared to the wavelengths of light. The absorbing and scattering particles should therefore be relatively far from each other in respect to the wavelength of light, as fibers in paper for example. The model assumes Lambertian illumination and is not adapted in case of collimated illumination. It also assumes strong, isotropic scattering in a homogenous semi-infinite medium. It does not apply with media whose scattering and absorption coefficient vary locally at mesoscopic scale, for example halftone ink dots in paper.
The $K$ and $S$ values for which accurate predictions may be expected from the model is often discussed. In many cases, in particular when the concentration in either absorbing particles or scattering particles is increases, the other one remaining constant, one observes that both $K$ and $S$ parameters vary, whereas only one of them should vary [82]. This leads to the conclusion that the two coefficients are interdependent. This is the reason why in many applications one rather consider the $K/S$ ratio, closely related to the reflectance of an infinitely thick medium according to Eq. (112), instead of separate $K$ and $S$ parameters. Lastly, anisotropy of the propagating fluxes is a main concern of the Kubelka-Munk theory. Nobbs [83] noticed that the assumption of uniformly diffuse flux propagating in opposite directions is valid only for nonabsorbing media, otherwise the fluxes become anisotropic and different absorption and scattering coefficients should be defined for each propagation direction. Anisotropic scattering is modeled by Monte-Carlo simulations or extensions of the Kubelka-Munk theory such as the multi-flux model developed by Mudgett and Richards [67], the four-flux model by Maheu et al. [84].

9. DIFFUSING MULTILAYERS

The Kubelka-Munk model describes the reflection and transmission of light by a uniform diffusing layer whose scattering and absorption coefficients are constant over the whole thickness. In 1954, Kubelka [77] proposed an extension for stacked diffusing layers describing flux transfers between the different layers. The layers are assumed to be isotropic. However, in contrast with the Kubelka-Munk model, they may be non-symmetric, i.e. their scattering and absorption coefficients may vary along the normal of the layer. Their two faces may therefore have different reflectances. The Fresnel reflections at the interfaces are not taken into account in the model, but a correction similar to the Saunderson correction for the Kubelka-Munk model can easily be introduced.

Throughout this section, the following notations will be used: the layers are labeled with a number $k = 1, 2, \ldots$ incremented in the order of superposition. The reflectances of their front face (front reflectance) and back face (back reflectance) are denoted respectively as $\rho_k$ and $\rho'_k$. Their transmittances are denoted as $\tau_k$ and $\tau'_k$ when light transits from front to back (downward transmittance), respectively from back to front (upward transmittance). Downward and upward fluxes are respectively denoted by letters $I$ and $J$, with an index $k = 0, 1, \ldots$ corresponding to their position in the multilayer: index 0 when the flux is in the surrounding medium at the front side, index 1 for the flux is between layers 1 and 2, and so on.
9.1. Kubelka’s compositional formulas

Let us consider the most general case of a stack of non-symmetric layers. Fig. 28 illustrates the flux transfers between two layers. By describing how the fluxes contribute to each other, we obtain the four following relations:

\[ J_0 = \rho_1 I_0 + \tau'_1 J_1 \]  
(123)

\[ J_1 = \rho_2 I_1 + \tau_2' J_2 \]  
(124)

\[ I_1 = \tau I_0 + \rho'_1 J_1 \]  
(125)

\[ I_2 = \tau_2 I_1 + \rho'_2 J_2 \]  
(126)

Fig. 28: Kubelka’s model described the reflections and transmissions (dashed arrows) of Lambertian fluxes (white arrows) between nonsymmetric diffusing layers.

The combination of relations (124) and (125) provides the following expression for \( I_1 \)

\[ I_1 = \frac{\tau_1}{1 - \rho_1 \rho_2} I_0 + \frac{\rho'_1 \tau'_2}{1 - \rho_1 \rho_2} J_2 \]  
(127)

The combination of relations (123), (124) and (127) enables expressing the flux \( J_0 \) exiting at the front side as a function of the incident fluxes, \( I_0 \) and \( J_2 \)

\[ J_0 = \left( \rho_1 + \frac{\tau_1 \rho_2}{1 - \rho_1 \rho_2} \right) I_0 + \left( \frac{\tau'_1 \tau'_2}{1 - \rho_1 \rho_2} \right) J_2 \]  
(128)

Likewise, the combination of relations (126) and (127) provides the following expression for the flux \( I_2 \) exiting at the back side as a function of the incident fluxes

\[ I_2 = \left( \frac{\tau_1 \tau_2}{1 - \rho_1 \rho_2} \right) I_0 + \left( \rho'_2 + \frac{\tau_2 \rho'_1}{1 - \rho_1 \rho_2} \right) J_2 \]  
(129)

The bracketed terms in Eq. (128) correspond to the front reflectance \( \rho_{12} \) and upward transmittance \( \tau'_{12} \) of the bilayer; the ones in Eq. (129) correspond to the downward
transmittance $\tau_{12}$ and the back reflectance $\rho'_{12}$. One retrieves Kubelka’s composition formulas:

$$\rho_{12} = \rho_1 + \frac{\tau_1 \tau_2 \rho_2}{1 - \rho' \rho_2}, \quad (130)$$

$$\tau_{12} = \frac{\tau_1 \tau_2}{1 - \rho' \rho_2}, \quad (131)$$

$$\rho'_{12} = \rho'_2 + \frac{\tau_2 \tau_2 \rho'_1}{1 - \rho' \rho_2}, \quad (132)$$

$$\tau'_{12} = \frac{\tau'_1 \tau'_2}{1 - \rho' \rho_2}. \quad (133)$$

### 9.2. Composition of the Kubelka-Munk reflectance and transmittance expressions

Kubelka’s composition formulas are compatible with the Kubelka-Munk reflectance and transmittance expressions expressed as functions of the scattering coefficient $S$, the absorption coefficient $K$ and the layer thickness $h$:

$$\rho_h = \frac{\sinh (bSh)}{a \sinh (bSh) + b \cosh (bSh)} \quad (134)$$

and

$$\tau_h = \frac{b}{a \sinh (bSh) + b \cosh (bSh)} \quad (135)$$

with

$$a = (K + S) / S \quad \text{and} \quad b = \sqrt{a^2 - 1} \quad (136)$$

If one superposes two layers made of the same material with respective thicknesses $h$ and $h'$, one obtains a layer of thickness $h'' = h + h'$ whose reflectance is given by Eq. (130) by setting $\rho_1 = \rho_1 = \rho_h$, $\tau_1 = \tau_1 = \tau_h$, $\rho_2 = \rho_2 = \rho_{h'}$ and $\tau_2 = \tau_2 = \tau_{h'}$. Thanks to the classical formulas of hyperbolic trigonometry [85], one may verify that

$$\rho_{h''} = \rho_h + \frac{\tau_h^2 \rho_{h'}}{1 - \rho_h \rho_{h'\prime}} = \frac{\sinh (bSh'\prime)}{a \sinh (bSh'\prime) + b \cosh (bSh'\prime)} \quad (137)$$

and

$$\tau_{h''} = \frac{\tau_h \tau_{h'\prime}}{1 - \rho_h \rho_{h'\prime}} = \frac{\sinh (bSh'\prime)}{a \sinh (bSh'\prime) + b \cosh (bSh'\prime)}, \quad (138)$$
9.3. Nonpolarity of transmittance

Kubelka stated that strongly scattering layers have identical upward and downward transmittances even when the scattering and absorbing properties of the medium vary according to the depth. This 'nonpolarity principle' is also true for diffusing multilayers. In order to put this property into evidence, let us consider two layers made of homogenous medium. According to the notations introduced above, we have $\rho_1 = \rho_1'$, $\tau_1 = \tau_1'$, $\rho_2 = \rho_2'$ and $\tau_2 = \tau_2'$. Bilayer’s upward and downward transmittances given by equations (131) and (133) are identical, while the front and back reflectances remain different.

9.4. Transfer matrices

Several methods enable obtaining Kubelka's composition formulas. A classical one consists in considering one incident flux, e.g. $I_0$, and summing up the different flux components exiting the bilayer at the front and back sides. The obtained sums are geometrical series which can be reduced to close form formulas. When the number of layers increases, the geometrical series are tedious to reduce if one considers all layers simultaneously, but one can first consider the two first layers, then consider this bilayer together with the third layer and so on. An alternative, matrix methods enable considering all layers at the same time. One of them relies on Markov chains modeling the flux transfers as transition probabilities and takes benefit of the Markov matrix formalism [86]. The size of the matrix varies linearly with the number of layers. In the present paper, we propose to develop a second method based on matrices with fixed size, called transfer matrices.

Let us first rearrange equations (123) and (125) in such manner as to express $I_0$ and $J_0$ as functions of $I_1$ and $J_1$

$$I_0 = \frac{1}{\tau_1'} I_1 - \frac{\rho_1'}{\tau_1} J_1$$

$$J_0 = \frac{\rho_1}{\tau_1} I_1 + \frac{\tau_1 \tau_1' - \rho_1 \rho_1'}{\tau_1} J_1$$

(139)

This system of equations can be written into a matrix form

$$\begin{pmatrix} J_0 \\ I_0 \end{pmatrix} = M_1 \begin{pmatrix} J_1 \\ I_1 \end{pmatrix}$$

(140)

where

$$M_1 = \frac{1}{\tau_1} \begin{pmatrix} 1 & -\rho_1' \\ \rho_1 & \tau_1 \tau_1' - \rho_1 \rho_1' \end{pmatrix}$$

(141)

is the transfer matrix attached to the first layer. By following this reasoning line for the second layer from equations (124) and (126), we obtain the transfer matrix of the second layer.
\[ M_2 = \frac{1}{\tau_2} \begin{pmatrix} 1 & -\rho'_2 \\ \rho_2 & \tau_2 \tau'_2 - \rho_2 \rho'_2 \end{pmatrix} \]  

(142)

On the one side, we have:

\[ \begin{pmatrix} J_0 \\ I_0 \end{pmatrix} = M_1 \begin{pmatrix} J_1 \\ I_1 \end{pmatrix} = M_1 M_2 \begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \]  

(143)

On the other side, by writing equations (128) and (129) under a matrix form, we have:

\[ \begin{pmatrix} J_0 \\ I_0 \end{pmatrix} = M_{12} \begin{pmatrix} J_2 \\ I_2 \end{pmatrix} \]  

(144)

with

\[ M_{12} = \frac{1}{\tau_{12}} \begin{pmatrix} 1 & -\rho'_{12} \\ \rho_{12} & \tau_{12} \tau'_{12} - \rho_{12} \rho'_{12} \end{pmatrix} \]  

(145)

and with \( \rho_{12}, \tau_{12}, \rho'_{12} \) and \( \tau'_{12} \) given by equations (130) to (133). The identification of equations (143) and (144) shows that \( M_{12} \) is the product of \( M_1 \) and \( M_2 \). Hence, the transfer matrix of a bilayer can be obtained by multiplying the transfer matrices of the two layers. The multiplicativity of transfer matrices is true whatever the number of layers is. Note that the left-to-right position of the matrices in the product reproduces the front-to-back position of the corresponding layers, i.e. for example for \( k \) layers,

\[ M_{123...k} = M_1 M_2 M_3...M_k \]  

(146)

Transfer matrices all have the same structure, i.e. reflectances and transmittances are arranged in the same way as in Eq. (141), (142) or (145). From a given transfer matrix \( M = \{m_{ij}\} \), one retrieves the front reflectance \( \rho \), back reflectance \( \rho' \), downward transmittance \( \tau \) and upward transmittance \( \tau' \) in the following way:

\[ \begin{align*}
\rho &= m_{21}/m_{11} \\
\tau &= 1/m_{11} \\
\rho' &= -m_{12}/m_{11} \\
\tau' &= \text{det}(M)/m_{11}
\end{align*} \]  

(147)

where the symbol ‘det’ denotes the determinant of the matrix.

Similar matrices, often called "transfer-matrices" in reference to the scattering matrices and scattering transfer matrices in electronics which also have similar properties, are used in wave optics to model the propagation of electric fields [87] or incoherent light beams [88, 90] in thin multilayers.
9.5. Discrete derivation of the Kubelka-Munk formulas

In its classical formulation, the Kubelka-Munk model relies on a linear system of differential equations, therefore on a continuous approach. We can also retrieve the Kubelka-Munk reflectance and transmittance expressions by following a discrete approach. The layer of thickness \( h \) is subdivided into \( n \) sublayers with identical thickness \( h/n \) which becomes infinitesimal as \( n \) grows to infinity. According to the Kubelka-Munk model, such infinitesimal sublayers absorb and backscatter fractions of incident flux being proportional to the sublayer's thickness. Since the reflectance \( \rho_{h/n} \) of one sublayer comes only from scattering, we have:

\[
\rho_{h/n} = Sh/n
\]

(148)

The transmittance \( \tau_{h/n} \) corresponds to the fraction of flux that is neither reflected or absorbed, i.e.

\[
\tau_{h/n} = 1 - (K + S)h/n = 1 - aSh/n
\]

(149)

with \( a = (K+S)/S \).

From these elemental reflectances and transmittances, let us calculate the reflectance and transmittance of the whole layer using the matrix method. The transfer matrix attached to one sublayer is

\[
M_{h/n} = \frac{1}{\tau_{h/n}} \begin{pmatrix}
1 & -\rho_{h/n} \\
-\rho_{h/n} & A_{h/n}
\end{pmatrix}
\begin{pmatrix}
1 & -Sh/n \\
Sh/n & 1 - 2aSh/n + \epsilon
\end{pmatrix}
\]

(150)

with \( A_{h/n} = \tau_{h/n}^2 - \rho_{h/n}^2 = 1 - 2aSh/n + (a^2 - 1)(Sh/n)^2 \). From a physical point of view, the term \( \epsilon = (a^2 - 1)(Sh/n)^2 \) corresponds to the second order scattering within the sublayer, i.e. the portion of light that is scattered twice before being reflected, transmitted or absorbed. This term tends to 0 as \( n \) tends to infinity. We may conclude that the Kubelka-Munk model ignores the second order scattering and describes the behavior of light at a scale where the medium is almost nonscattering.

Since the whole layer corresponds to the superposition of \( n \) identical sublayers, its transfer matrix is given by

\[
M_h = \left( M_{h/n} \right)^n
\]

(151)

Since Eq. (151) holds for any nonzero integer \( n \), it also holds as \( n \) grows to infinity

\[
M_h = \lim_{n \to \infty} \left( M_{h/n} \right)^n
\]

(152)

Let us write matrix \( M_{h/n} \) as follows

\[
M_{h/n} = \frac{1}{1-aSh/n} \left[ I_4 + \frac{1}{n} A \right]
\]

(153)
where $I_4$ is the $2 \times 2$ identity matrix and $A$ is the matrix defined as

$$A = \begin{pmatrix} 0 & -Sh \\ Sh & -2aSh + \varepsilon \end{pmatrix}$$

(154)

where the term $\varepsilon$ can be neglected. We will use a classical limit of the exponent function

$$\lim_{n \to \infty} \left(1 - \frac{x}{n}\right)^n = e^{-x}$$

that can also be extended to the matrix exponential in the case of a matrix $B$:

$$\lim_{n \to \infty} \left( I + \frac{1}{n} B \right)^n = \exp(B)$$

This yields

$$M_h = \lim_{n \to \infty} \left( M_{h/n} \right)^n = e^{aSh} \exp(A)$$

(155)

The diagonalization of matrix $\exp(A)$ is obtained through the diagonalization of $A$

$$\exp(A) = \begin{pmatrix} a - b & -1 \\ a + b & -1 \end{pmatrix}^{-1} \cdot \text{diag} \left( e^{-(a+b)Sh}, e^{-(a-b)Sh} \right) \cdot \begin{pmatrix} a - b & -1 \\ a + b & -1 \end{pmatrix}$$

(156)

$$= e^{-aSh} \begin{pmatrix} b \cosh(bSh) + a \sinh(bSh) & -\sinh(bSh) \\ \sinh(bSh) & b \cosh(bSh) - a \sinh(bSh) \end{pmatrix}$$

with $b = \sqrt{a^2 - 1}$. Therefore, the transfer matrix $M_h$ is

$$M_h = \frac{1}{b} \begin{pmatrix} b \cosh(bSh) + a \sinh(bSh) & -\sinh(bSh) \\ \sinh(bSh) & b \cosh(bSh) - a \sinh(bSh) \end{pmatrix}$$

(157)

As expected, the inverse of the top-left entry and the ratio of the back-left entry to the top-left give respectively the Kubelka-Munk transmittance and reflectance expressions.

The use of infinitesimally thin sublayers for obtaining Kubelka–Munk expressions needs some comments. Usually, scattering is due to heterogeneities in the medium, e.g. particles, whose size cannot be assumed as infinitesimally small. According to the intrinsic properties of the diffusing medium, a model should be chosen for the description of the scattering of light by a single particle (single scattering model, such as Mie’s theory) or by collections of particles (multiple scattering model). It is possible to first determine the reflectance and the transmittance of an elementary sublayer made of this diffusing medium and, afterwards, to use the discrete two-flux model to consider various superposed sublayers, like in Melamed’s model for pigment slurries [63]. The discrete two-flux model should be used when the sublayer behaves as a perfect diffuser, with the assumption that the medium is intensely diffusing and that the sublayer has a minimal thickness, at least the size of an average particle.
The upward and downward fluxes are evaluated at discrete depths only, corresponding to multiples of the sublayer thickness. However, the equivalence that has been established between the continuous and the discrete models allows one to associate to the real diffusing medium an ‘imaginary’ medium; this medium is characterized by a scattering coefficient and an absorption coefficient such that the Kubelka–Munk model gives the same values for upward and downward fluxes at the discrete depths considered in the discrete model. At the intermediate depths, the value given by the continuous model corresponds to a mathematical interpolation.

### 9.6. Saunderson correction in multilayers: matrix method

In a diffusing multilayer, the interfaces between layers have no optical effect if the layers have the same refractive index. Otherwise, light reflections and refractions occur at the interfaces and modify the flux transfers between the layers. In a similar manner as Saunderson for the Kubelka-Munk model, we propose to include the interface effect into our matrix model.

![Kubelka’s model with the flux reflections and transmissions by the diffusing layers (dashed arrows) and the interfaces (solid arrows).](image)

Let us first recall from Section 4.6 that an interface receiving Lambertian light on its front face (refractive index $n_1$) and its back face ($n_2$) has the front reflectance reflectance $r_{12}$, the downward transmittance $t_{12}$, the back reflectance $r_{21}$, and the upward transmittance $t_{21}$ given by equations (46) to (51). These reflectances and transmittances can be easily introduced in Kubelka’s model by considering the superposition of layers and interfaces and multiplying their respective transfer matrices. The transfer matrices attached to the interfaces are similarly defined as the one of the layers:
\[ f_{12} = \frac{1}{t_{12}} \begin{pmatrix} 1 & -r_{21} \\ r_{12} & t_{12} t_{21} - r_{12} r_{21} \end{pmatrix} \] (158)

Since \( r_{21}, t_{12} \) and \( t_{21} \) can all be expressed as functions of \( r_{12} \) (see equations (43), (47) and (48) of Chapter 1), only this latter needs to be computed. Matrix \( f_{12} \) can be written as

\[ f_{12} = \frac{1}{n_2^2 (1 - r_{12})} \begin{pmatrix} n_2^2 & n_1^2 (1 - r_{12}) - 1 \\ n_2^2 r_{12} & (n_2^2 + n_1^2)(1 - r_{12}) - 1 \end{pmatrix} \] (159)

In order to illustrate the Saunderson correction for multilayers, let us consider two superposed layers with their three interfaces (Fig. 29). We assume that in each surrounding medium, the light source is Lambertian and the detector collects all exiting light. The layers have the transfer matrices \( M_1 \) and \( M_2 \) defined as in Eq. (141), and the three interfaces have the transfer matrices \( f_{01}, f_{12} \) and \( f_{23} \) defined as in Eq. (158). By multiplying these matrices according to the sequence of interfaces and layers in the multilayer, we obtain the transfer matrix:

\[ M = f_{01} M_1 f_{12} M_2 f_{23} \] (160)

The reflectances and transmittances of the multilayer can be deduced from the entries of \( M \) according to the formulas (147).

10. NONSCATTERING MULTILAYERS

A nonscattering multilayer is a stack of transparent, absorbing layers having possibly different refractive indices. Although nonscattering media do not reflect light by themselves, their interfaces are reflecting and a multiple reflection process takes place between them, similar to the one described in Section 5 within an absorbing plate. In order to easily obtain the reflectance and transmittance of many stacked layers, we introduce a matrix model similar to the one presented for diffusing multilayers, based on transfer matrices, with the difference that orientation and polarization of light must be taken into account.

10.1. Orientation and polarization of light

Due to the laws of reflection and refraction at flat interfaces (Snell's laws), the orientations of light in the multilayer are fully determined by the ray's initial orientation. These orientations, in their turn, determine the attenuation of light within the layers as well as the reflectivities of the interfaces. The multiple reflections of light must therefore be described by considering one incident angle. This rule will be referred to as the "Directionality principle". For every multilayer, there exist analytical reflectance and transmittance expressions which are
functions of the incident angle. Once having these expressions, they may be valued for a
given angle or integrated over a given set of angles.

Similar considerations apply with polarization because the reflectivity of the interfaces
depends on the polarization of light. If the incident light is linearly polarized in or
perpendicularly to the plane of incidence, this polarization is maintained in the whole multiple
reflection process and determines which Fresnel formula must be used for the reflectivities
contained within the multilayer's reflectance and transmittance expressions. In the case of
natural light, which is the type of light emitted by most light sources, one assumes that the
two linearly polarized lights flow in parallel, independently to each other and with equal
initial power (See Section 2.2). The multilayer's reflectance and transmittance is the average
of the reflectances, respectively transmittances, attached to the parallel (p) and perpendicular
(s) polarizations. This may be written by the following equation

\[ X = \frac{f (R_p) + f (R_s)}{2} \]  (161)

where \( X \) denotes a multilayer reflectance (or transmittance) for natural light, \( f \) the analytical
reflectance (respectively transmittance) expression being a function the Fresnel reflectivity
\( R_p \) for the parallel polarization or \( R_s \) the perpendicular polarization.

Nevertheless, many nonscattering materials such as polymers have a strong optical anisotropy
which decreases the degree of polarization of light. In this case, it is more exact to consider
that light remains unpolarized along the whole multiple reflection process and to insert the
Fresnel formula for unpolarized light (which is the average of the formulas for p- and s-
polarized lights) in the multilayers' reflectance and transmittance expressions. The multilayer
reflectance or transmittance \( X \) thus becomes

\[ X = f \left( \frac{R_p + R_s}{2} \right) \]  (162)

In the present section, we will consider this latter case. In addition to being simpler because it
prevents having to expand expressions for each polarization and to average them, this choice
is justified by the fact that polymers are used in our experimental applications. Nevertheless,
it is not an issue to follow one approach or the other one since the model in itself only
provides the function \( f \).

10.2. Matrix model for nonscattering multilayers

Let us now introduce de matrix model, by considering two superposed layers with refractive
indices \( n_1 \) and \( n_2 \) and normal transmittances \( t_1 \) and \( t_2 \) (Fig. 30). At the front and back sides,
the surrounding media have refractive indices \( n_0 \), respectively \( n_3 \). The incident light beam
comes from medium 0 at angle $\theta_0$. At each interface, the beam is split into reflected and refracted components propagating in the regular directions given by Snell’s laws. After splitting, each beam is then reflected and refracted each time it meets again an interface. It is obvious when looking at Fig. 30(a) that all light paths located in one given medium $i = 0, 1, 2$ or 3 form the same angle $\theta_i$ with the normal, which is related to $\theta_0$ by Snell’s sine law:

$$\theta_i = \arcsin \left( \frac{n_0 \sin \theta_0}{n_i} \right)$$

(163)

Fig. 30. (a) Multiple reflections of directional light in a multilayer composed of two nonscattering layers; (b) flux notation.

In each medium $i$, the components oriented downwards, represented in red in the Fig., are gathered as forming one flux $I_i(\theta_i)$. Likewise, the components directed upward, represented in green, form the flux $J_i(\theta_i)$. In the absorbing layers, the fluxes are attenuated by a factor $t_i^{-1/cos \theta_i}$ ($i = 1, 2$) due to absorption. Let us denote as $I_i'(\theta_i)$ and $J_i'(\theta_i)$ these fluxes once being attenuated:

$$I_i'(\theta_i) = t_i^{1/cos \theta_i} I_i(\theta_i)$$

$$J_i'(\theta_i) = t_i^{1/cos \theta_i} J_i(\theta_i)$$

(164)

The fluxes $I_0(\theta_0)$ and $J_0(\theta_0)$ striking the front interface contribute to the fluxes $J_i(\theta_0)$ and $I_i(\theta_0)$, either by reflection or by transmission. Using the same notation as in Section 4 for the Fresnel reflectivity and transmittivity, we obtain the following two equations

$$I_i(\theta_i) = T_{0i}(\theta_0) I_0(\theta_0) + R_{0i}(\theta_1) J_i'(\theta_i)$$

$$J_0(\theta_0) = R_{01}(\theta_0) I_0(\theta_0) + T_{10}(\theta_1) J_i'(\theta_i)$$

(165)
Equations (165) can be transformed in such manner as to express $I_0(\theta_0)$ and $J_0(\theta_0)$ as functions of $I_1(\theta_1)$ and $J_1(\theta_1)$, then converted into the following vector Eq.

$$\begin{pmatrix} I_0(\theta_0) \\ J_0(\theta_0) \end{pmatrix} = F_{01}(\theta_0) \begin{pmatrix} I_1(\theta_1) \\ J_1(\theta_1) \end{pmatrix}$$

(166)

where $F_{01}(\theta_0)$ is the transfer matrix of the interface, defined as

$$F_{01}(\theta_0) = \frac{1}{T_{01}(\theta_0)} \begin{pmatrix} 1 & -R_{10}(\theta_1) \\ R_{01}(\theta_0) & A_{01} \end{pmatrix}$$

(167)

with $A_{01} = T_{01}(\theta_0)T_{10}(\theta_1) - R_{01}(\theta_0)R_{10}(\theta_1)$. Since angles $\theta_0$ and $\theta_1$ are linked by Snell’s sine law, the reflectivities $R_{01}(\theta_0)$ and $R_{10}(\theta_1)$ are equal and the transmittivities $T_{01}(\theta_0)$ and $T_{10}(\theta_1)$ are both equal to $1 - R_{01}(\theta_0)$. All of the entries of the transfer matrix can therefore be expressed as functions of $\theta_0$:

$$F_{01}(\theta_0) = \frac{1}{1 - R_{01}(\theta_0)} \begin{pmatrix} 1 & -R_{01}(\theta_0) \\ R_{01}(\theta_0) & 1 - 2R_{01}(\theta_0) \end{pmatrix}$$

(168)

By following this reasoning line for the middle and back interfaces, we obtain the following vector equations

$$\begin{pmatrix} I_2(\theta_1) \\ J_2(\theta_1) \end{pmatrix} = F_{12}(\theta_1) \begin{pmatrix} I_1(\theta_1) \\ J_1(\theta_1) \end{pmatrix}$$

(169)

and

$$\begin{pmatrix} I_3(\theta_2) \\ J_3(\theta_2) \end{pmatrix} = F_{23}(\theta_2) \begin{pmatrix} I_2(\theta_2) \\ J_2(\theta_2) \end{pmatrix}$$

(170)

where $F_{12}(\theta_1)$ and $F_{23}(\theta_2)$ are the transfer matrices attached to the middle, respectively the back interfaces, defined as $F_{01}(\theta_1)$.

Regarding now the first layer, the incoming and outgoing fluxes related by Eq. (164) provide the following vector equation:

$$\begin{pmatrix} I_1(\theta_1) \\ J_1(\theta_1) \end{pmatrix} = L_1(\theta_1) \begin{pmatrix} I_1'(\theta_1) \\ J_1'(\theta_1) \end{pmatrix}$$

(171)

where

$$L_1(\theta_1) = \frac{1}{t_1'\cos \theta_1} \begin{pmatrix} 1 & 0 \\ 0 & t_1'' \cos \theta_1 \end{pmatrix}$$

(172)

is the transfer matrix of the nonscattering layer, independent of polarization of light. By following the same reasoning line for the second layer, we obtain the same Eq. with indices 2 in place of indices 1. Finally, we are interested in relating the incoming and outgoing fluxes
\( J_0(\theta_0), I_0(\theta_0), I_3(\theta_3), J_3(\theta_3) \) from which the reflectances and transmittances of the multilayer are defined. Thanks to the previous vector relations, one has:

\[
\begin{pmatrix}
I_0(\theta_0) \\
J_0(\theta_0)
\end{pmatrix} = F_{01}(\theta_0) L_1(\theta_1) F_{12}(\theta_1) L_2(\theta_2) F_{23}(\theta_2) \begin{pmatrix}
I_3(\theta_3) \\
J_3(\theta_3)
\end{pmatrix}
\]

(173)

Thus, the multilayer’s transfer matrix is the product of the transfer matrices attached to the interfaces and the layers. Their multiplication order, from left to right, reproduces the sequence the corresponding interfaces and layers from front to back. The multilayer’s reflectances and transmittances are deduced from its entries as indicated by equations (147). All angles are functions of the incident angle \( \theta_0 \) given by Eq. (163). If the light comes from medium 3, they can also be all expressed as functions of \( \theta_3 \), i.e. \( \theta_i = \arcsin(n_i \sin(\theta_3)/n_i) \) for every \( i \).

Transfer matrices have similar structure as the ones for diffusing layers: the front and back reflectances and the upwards and downwards transmittances are arranged in the same way and may be retrieved by similar operations on the matrix entries as in equations (147). The matrix method applies with any stack of nonscattering layers, for any incident angle. The orientation of light in the different layers is automatically embodied in the transfer matrices provided they are defined and multiplied correctly. However, we have assumed that the light can cross the whole multilayer, which is certainly not the case when a total reflection occurs at a certain interface. The following section explains how total reflections can be automatically treated.

**10.3. Total reflections**

In each layer \( i \), both upward and downward fluxes are oriented according to an angle \( \theta_i \) related to the incident angle \( \theta_0 \) by relation Snell's sine law:

\[
\theta_i = \arcsin\left(n_i \sin(\theta_0)/n_i\right)
\]

(174)

At normal incidence, light is perpendicular to all layers, i.e. \( \theta_0 = \theta_1 = \theta_2 = ... = 0 \). As \( \theta_0 \) moves away from the normal, all angles \( \theta_i \) also move away from the normal. If one of the layers' indices is such that \( n_0 \sin(\theta_0)/n_i > 1 \), the corresponding angle \( \theta_i \) has no real value anymore, which means that light does not enter the layer \( i \), i.e. it is totally reflected by the \( i \)th interface.

Fig. 31 illustrates the fact that different interfaces become totally reflecting when the incident angle increases. As long as \( n_0 \sin(\theta_0) \) is superior to each of the multilayer refractive indices \( n_1, n_2 \) and \( n_3 \), i.e., in the present case, as long as \( \theta_0 \) is inferior to \( 41.8^\circ = \arcsin(1/1.5) \), the light pencil can cross the multilayer; the incident rays that are not subject to total reflection are contained into a cone of half-angle:
For larger $\theta_0$, $n_0 \sin \theta_0$ becomes first superior to $n_3$, then superior to $n_2$, yielding consequently a total reflection at respectively the acetate-air interface and the glass-water interface.

![Fig. 31. Transmission and total reflection of directional light in a multilayer for different incident angles.](image)

Regarding the transfer matrices, the ones attached to the interfaces have their entries divided by the Fresnel transmittivity, which becomes zero in case of total reflection. This induces a serious computational problem. In order to prevent it, we recommend using an alternative definition for the transfer matrices where the downward transmittance is incorporated into the matrix in a third row and third column:

$$
M = \begin{pmatrix}
1 & -\rho' & 0 \\
\rho & \tau' - \rho \rho' & 0 \\
0 & 0 & \tau
\end{pmatrix}
$$

These $3 \times 3$ matrices can be multiplied exactly in the same way as the $2 \times 2$ transfer matrices without any chance of division by zero. The upper reflectance $\rho$, back reflectance $\rho'$, downward reflectance $\tau$ and upward transmittance $\tau'$ are deduced from the entries $m_{ij}$ ($i,j = 1,2,3$) of the $3 \times 3$ matrix in the following way:

$$
\rho = m_{21} / m_{11} \\
\tau = m_{33} / m_{11} \\
\rho' = -m_{12} / m_{11} \\
\tau' = \left[ \det(M) / m_{11} \right] / m_{33}^2
$$

Let us examine what happens in the matrix product of Eq. (173) when a total reflection occurs at the middle interface. The transfer matrix of the middle interface, $F_{12}(\theta_1)$, is similar to the one given in Eq. (168) with the Fresnel reflectivity $R_{12}(\theta_1)$ equal to 1. After conversion into the $3 \times 3$ matrix format, we have
\[
\mathbf{F}_{12}(\theta_i) = \begin{pmatrix}
1 & -1 & 0 \\
1 & -1 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] 

(178)

With this matrix \( \mathbf{F}_{12}(\theta_i) \), the product of matrices in Eq. (173) provides the following front reflectance for the multilayer, which contains no terms relative to the second layer or the back interface:

\[
R_{0123}(\theta_0) = R_{01}(\theta_i) + \frac{T_{01}(\theta_i)T_{01}(\theta_0)t_i^{2/\cos\theta_i}}{1 - R_{01}(\theta_i)t_i^{2/\cos\theta_i}}
\] 

(179)

The fact that no light enters the second layer is therefore automatically taken into account by the model. The downward transmittance is zero. The back reflectance and the upward transmittance, in this case, make no sense anymore.

The 2×2 transfer matrices previously introduced have the advantage to relate flux vectors and make sense when the upward and downward fluxes are nonzero. This is the reason why we will continue to use them throughout this paper despite the numerical problem evoked above. In computational programs, however, we strongly recommend using the 3×3 transfer matrices.

### 10.4. Nonpolarity of directional transmittance

We observe from equations (168) and (172) that the transfer matrices of interfaces and nonscattering layers have a determinant equal to 1. Since the transfer matrix \( \mathbf{M} \) of a multilayer is a product of these elementary transfer matrices, its determinant is also equal to 1. Consequently, by denoting as \( m_{11} \) the top-left entry of \( \mathbf{M} \), we have the following equality for every nonscattering multilayer:

\[
\det(\mathbf{M})/m_{11} = 1/m_{11}
\] 

(180)

which means, in accordance with equations (147), that the upward and downward transmittances are identical. This property, which echoes the nonpolarity of transmittance stated by Kubelka for diffusing multilayers, is true for directional light in absence of total reflection. It is therefore always true at normal incidence. However, the nonpolarity property does not apply with Lambertian transmittances. The interface between air and glass (refractive index 1.5) is a good example since its Lambertian transmittance is equal to 0.9 when the light comes from air and equal to 0.4 when the light comes from the glass (see Section 4.6).

### 10.5. Radiance transmission and Lambertian illumination

The directional reflectance (transmittance) given by the multilayer model is a ratio of reflected (respectively transmitted) to incident directional fluxes. We are now interested in the
reflection and transmission of radiance. By definition, radiance is a density of flux per unit apparent area and unit solid angle. If the incident and emerging light pencils fulfill the same solid angle, the ratio of the radiances is equal to the ratio of the fluxes. This is the case for the reflection: the ratio of reflected to incident radiances is therefore equal to the directional reflectance. Regarding the transmission, the refractions may induce a changing of solid angle between the transmitted flux and the incident flux. According to whether the transmitted flux is spread to a larger solid angle or concentrated into a smaller solid angle, the transmitted radiance will be lower, respectively higher than the incident radiance multiplied by the directional transmittance of the multilayer. The changing of solid angle is taken into account by a geometrical factor which depends only on the refractive indices of the two surrounding media. We have already encountered this factor in Eq. (40), in the context of the transmission of radiance through an interface. This result can be directly generalized to nonscattering multilayers: by denoting as \( L_0(\theta_0) \) the incident radiance coming at angle \( \theta_0 \) from medium 0 of refractive index \( n_0 \), and as \( L_k(\theta_k) \) the transmitted radiance exiting at angle \( \theta_k \) into medium \( k \) of refractive index \( n_k \), one has

\[
L_k(\theta_k) = \left(\frac{n_k}{n_0}\right)^2 T_{012\ldots0}(\theta_0) L_0(\theta_0)
\]

where \( \left(\frac{n_k}{n_0}\right)^2 \) is the geometrical factor accounting for the changing of solid angle and \( T_{012\ldots0}(\theta_0) \) denotes the transmittance of the multilayer for natural light.

When the incident light is Lambertian, each incident ray follows its own path within the multilayer independently of each other (directionality principle). The Lambertian reflectance and Lambertian transmittance account for the reflection, respectively the transmission of all these rays and are expressed by an integral of the form

\[
f = \int_{\theta=0}^{\pi/2} F(\theta) \sin 2\theta d\theta ,
\]

where \( F \) denotes the directional reflectance or transmittance of the multilayer.

10.6. Stacks of films at normal incidence

In order to illustrate the application of the matrix model, we propose to model the spectral reflectance and transmittance of stacks of identical nonscattering acetate films, whose individual reflectance \( R \) and transmittance \( T \) have been measured in the normal to the surface. Both \( R \) and \( T \) embody the multiple reflections between the film's surfaces. Since the Fresnel formulae are identical at normal incidence, \( R \) and \( T \) are identical for parallely and perpendicularly polarized lights, thereby for natural light. The transfer matrix of one film, defined as
\[ \mathbf{Q} = \frac{1}{T} \begin{pmatrix} 1 & -R \\ R & T^2 - R^2 \end{pmatrix} \]  

(183)

can be transformed by diagonalization as

\[ \mathbf{Q} = \frac{1}{T} \mathbf{E} \cdot \begin{pmatrix} 1-(a+b)R & 0 \\ 0 & 1-(a-b)R \end{pmatrix} \cdot \mathbf{E}^{-1} \]  

(184)

with

\[ \mathbf{E} = \begin{pmatrix} a-b & a+b \\ 1 & 1 \end{pmatrix} \]  

(185)

\[ a = \frac{1+R^2-T^2}{2R} \]  

(186)

and

\[ b = \sqrt{a^2 - 1} \]  

(187)

When several films are stacked, unless they are pasted with a fluid, there is no optical contact between them: they are separated by thin air slices. The relative index of the surfaces is therefore not modified and each film in the stack is characterized by the transfer matrix \( \mathbf{Q} \) given by Eq. (183). The transfer matrix of \( k \) films, obtained by multiplying these \( k \) identical transfer matrices, is therefore \( \mathbf{Q}^k \). Thanks to the matrix diagonalization (184), one has

\[ \mathbf{Q}^k = \frac{1}{T^k} \mathbf{E} \cdot \begin{pmatrix} [1-(a+b)R]^k & 0 \\ 0 & [1-(a-b)R]^k \end{pmatrix} \cdot \mathbf{E}^{-1} \]  

(188)

After computation, the bottom-left entry of this matrix divided by its top-left entry provides the following closed-form reflectance expression:

\[ R_k = \frac{1}{a-b} \left( \frac{2}{1-\frac{1-(a+b)R}{1-(a-b)R}} \right)^{k} \]  

(189)

and the inverse of its top-left entry provides the following transmittance formula:

\[ T_k = \frac{2bT^k}{(a+b)[1-(a-b)R]^k-(a-b)[1-(a+b)R]^k} \]  

(190)

As \( k \) increases, the stack contains more layers and interfaces and thus becomes at the time more absorbing and more reflecting. The reflectance increases until a limit value corresponding to the reflectance of an infinite stack, denoted as \( R_\infty \). Since the term raised at
the power \( k \) in Eq. (189) is smaller than 1, it tends to zero. The infinite stack reflectance is therefore

\[
R_{\infty} = \frac{1}{a + b} = a - b
\]  

(191)

Oppositely, since the photons have less chance to cross the multilayer without being absorbed or back-reflected by interfaces, the transmittance decreases. It approaches zero more rapidly as the layers are more absorbing.

**10.7. Stacks of films at oblique incidence**

From the film's reflectance \( R \) and transmittance \( T \) measured at normal incidence, we now want to predict the reflectance \( R_k(\theta) \) and transmittance \( T_k(\theta) \) of \( k \) stacked films at any incidence \( \theta \). This can be achieved if the normal transmittance \( t \) of the film bulk and its refractive index are known, by multiplying the transfer matrices of the interfaces and film layers with the appropriate angles.

Let us define the transfer matrices \( F_{01}(\theta) \) and \( F_{10}(\theta_1) \) attached to the front and back interfaces of one film according to Eq. (167), with \( \theta_1 = \arcsin(\sin \theta/n_1) \) the angle at which the incident light is refracted into the film, and the transfer matrix \( L(\theta_1) \) attached to the film bulk layer with normal transmittance \( t \) according to Eq. (172). The transfer matrix attached to the film is

\[
Q(\theta) = F_{01}(\theta)L(\theta_1)F_{10}(\theta_1) = \frac{1}{T(\theta)} \begin{pmatrix} 1 & -R(\theta) \\ R(\theta) & T(\theta)^2 - R(\theta)^2 \end{pmatrix}
\]  

(192)

where \( R(\theta) \) and \( T(\theta) \) are the film's reflectance and transmittance at incidence \( \theta \), deduced from \( \mathbf{M} \) thanks to the relations (147):

\[
R(\theta) = 2 \frac{\left[1 - R_{01}(\theta)\right]^2 R_{01}(\theta) t^2 \sqrt{1 - (\sin \theta/n)^2}}{1 - R_{01}^2(\theta) t^2 \sqrt{1 - (\sin \theta/n)^2}}
\]  

(193)

and

\[
T(\theta) = \frac{\left[1 - R_{01}(\theta)\right]^2 t^2 \sqrt{1 - (\sin \theta/n)^2}}{1 - R_{01}^2(\theta) t^2 \sqrt{1 - (\sin \theta/n)^2}},
\]  

(194)

The same expressions are derived from geometrical series in Section 5, except that here, as explained in Section 4.1, the Fresnel reflectivities are defined for natural light whereas they are defined for \( p- \) or \( s- \) polarization in Section 4. The fact that most polymer films are optically anisotropic, whereas the glass plate considered in Section 5 has more chance to be isotropic, explains this difference. However, there is no difference anymore at normal
incidence since the reflectivities for linearly polarized light and for natural light are equal. The formula (65) giving the normal transmittance \( t \) from the film's transmittance \( T \) measured at normal incidence, is therefore valid in both cases:

\[
t = \sqrt{\frac{64n_1^4 + (1-n_1^2)^2}{(1-n_1^2)^4}} \frac{T^2}{T} - 8n_1^2
\]  
(195)

This formula is useful to determine the normal transmittance of the film bulk when it is unknown.

Eq. (192) can be easily extended to a stack of \( k \) films. Since all films are separated by thin air slices, they receive light at the angle \( \theta \) and are therefore all represented by the transfer matrix \( Q(\theta) \) given in (192). The transfer matrix of the pile is therefore

\[
Q_k(\theta) = Q^k(\theta) = \left[ F_{01}(\theta) L_1(\theta_1) F_{10}(\theta_1) \right]^k
\]  
(196)

By analogy with the equations previously derived for the normal incidence, we conclude that the expressions for the pile's reflectance and transmittance are similar the ones given in Eq. (189), respectively in Eq. (190), with the \( R(\theta) \) and \( T(\theta) \) expressed by equations (61) and (194) in place of \( R \) and \( T \).

10.8. Stacks of films pasted with liquid

We can modify the relative index of the film surfaces by pasting them with a clear liquid \( n_2 \) (Fig. 32). With two films, the multilayer is composed of (from front to back): an air-acetate interface [transfer matrix \( F_{01}(\theta) \) given by Eq. (168)], a film bulk layer [transfer matrix \( L(\theta_1) \) given by Eq. (172)], an acetate-liquid interface [transfer matrix \( F_{12}(\theta_1) \)], a slice of nonabsorbing liquid (identity matrix), a liquid-acetate interface [transfer matrix \( F_{21}(\theta_2) \)], a second film bulk layer (same transfer matrix as the first one) and an acetate-air interface [transfer matrix \( F_{10}(\theta_1) \)]. The transfer matrix attached to the multilayer is

\[
S_{k=2}(\theta_0) = F_{01}(\theta_0) L(\theta_1) F_{12}(\theta_1) F_{21}(\theta_2) L(\theta_1) F_{10}(\theta_1)
\]  
(197)

Likewise, \( k \) identical films pasted with the same medium have the transfer matrix:

\[
S_k(\theta_0) = F_{01}(\theta_0) L_1(\theta_1) F_{12}(\theta_1) F_{21}(\theta_2) L_1(\theta_1) F_{10}(\theta_1)
\]  
(198)

The reflectance and the transmittance of the stacks are deduced from the entries of the transfer matrices as indicated by equations (147). Their analytical expressions are too complex to be reproduced here, but their numerical computation doesn’t pose any difficulty. Since the surrounding medium, i.e. air, has a lower refractive index than the acetate and the liquid, no total reflection can occur whatever the incident angle is.
We can also imagine a stack where the films are pasted with a fluid of the same refractive index \( n_2 = n_1 = 1.54 \). In this case, the acetate-fluid interfaces have no optical effect: every stack is equivalent to one acetate layer with normal transmittance \( t_\lambda^k \), \( k \) being the number of films. The reflectance and transmittance of this layer are similarly expressed as those of a single film, with \( t_\lambda \) being replaced by \( t_\lambda^k \).

### 10.9. Experimental testing

In order to illustrate the model, we propose a real experiment carried out with stacks of blue acetate films with interstices filled by air (\( n_2 = 1 \)) or by liquid (\( n_2 = 1.4 \) or \( n_2 = 1.54 \)), illuminated at normal incidence. The predicted spectra, plotted in Fig. 33 for different numbers of films, are provided by the equations (189) and (190) in the case of air interstices, and by the reflectance and transmittance expressions issued from the transfer matrix given in Eq. (198) in the case of fluid-filled interstices. We observe fairly good accordance between predictions and measurements in transmittance mode: the \( \Delta E_{94} \) values were below 0.5 for the films separated by air (2 to 5 films) and below 0.3 for films separated by the liquid of index 1.4. In reflectance mode, we obtained slightly higher \( \Delta E_{94} \) values comprised between 1.05 and 1.4 for the films separated by air. Interferences within the thin air slices may be at the origin of this loss of accuracy.

Let us analyze the spectrum variation in respect of the different parameters of the model. The interaction between light and stacks may be summarized as the combination of an absorption phenomenon within the layers and of a back-reflection phenomenon at the interfaces. The absorption phenomenon attenuates both the reflectance and the transmittance in a wavelength dependent manner. The back-reflection phenomenon tends to increase the reflectance and to decrease the transmittance in a proportion which depends on the reflectivity of the interfaces but not on the wavelength. The two phenomena are amplified when one film is added to the stack. This yields spectral variations depending on the wavelength. The transmittance of the stacks decreases as the number of films is incremented. In the spectral domain where the acetate is the more absorbing (570 – 680 nm), it strives rapidly towards zero. Outside this domain, the transmittance decreases more slowly, in proportion to the reflectivity of the added acetate-liquid interfaces.
Regarding the reflectance, it is increased by adding films provided the reflectivity of the interfaces is sufficiently high to compensate loss due to absorption. The interfaces have the highest reflectivity in the stacks where the films are separated by air. The reflectance decreases in all spectral bands. When the films are separated by the liquid of refractive index 1.4, the reflectance increases only in the spectral bands where the acetate is less absorbing (440 – 510 nm) and decreases in the other spectral bands. We notice a perfect compensation at 440 nm and 510 nm. When the refractive index of the pasting medium matches the one of the films, the interfaces in the multilayer are not reflecting, except the front and back interfaces. The front interface reflects an achromatic component (reflectance $r_0$) and the multiple reflection process between the front and back interfaces provides a colored component which decreases as the number of films, thereby the global absorption, increases. As the number of
films grows to infinity, the light entering the stack is completely absorbed before reaching the back interface; the reflectance of the infinite pile is therefore equal to the achromatic reflectance of the front interface, i.e. $r_0$.

In the complete study presented in [89], color variations of the stacks are analyzed in respect to the number of films and the refractive index of the medium between the films. When the number of films is incremented, the increase or decrease of spectrum leads to similar increase, respectively decrease of lightness. Saturation has a more complex evolution linked to the different spectral variations in the spectral domains where light is most and less absorbed. When adding one film increases the difference between the maximum and the minimum of the spectrum, saturation also increases (see for example the transmittances spectra of 1 to 6 films separated by air or by liquid, as well as the reflectance spectra of films separated by air). Oppositely, saturation is decreased when this difference decreases (see the spectral transmittance of stacks containing more than 6 films separated by air, or the spectral reflectance of stack containing more than 4 films pasted by the liquid with index 1.54). In every case, there exist an optimal number of films for which saturation is maximal.

11. NONSCATTERING MULTILAYER ON TOP OF A DIFFUSING BACKGROUND

In the two previous sections, we developed two-flux models adapted to either diffusing or nonscattering multilayers. We now propose to deal with stacks of both diffusing and nonscattering layers. Modeling their reflectance and transmittance by a two-flux approach, i.e. by the multiplication of transfer matrices, is possible provided the directionality principle is respected concerning the nonscattering elements (see Section 3.1, and for more details concerning the applicability of two-flux models in multilayer, see Ref. [86]).

11.1. General method

The directionality principle allows multiplying the transfer matrices of nonscattering layers and/or flat interfaces only if light is directional. The transfer matrix resulting from the multiplication provides analytical reflectance and transmittance expressions being functions of the incident angle. These latter can be integrated over the hemisphere if diffuse light comes, for example, from a diffuse light source or a diffusing layer. Consequently, every sequence of nonscattering layers and flat interfaces must be considered as being one nonscattering element whose transfer matrix is defined in respect to the geometries of illumination and observation on its two sides. In order to be sure that the directionality principle is respected, we recommend following the procedure in three steps for each sequence of consecutive nonscattering elements in a multilayer:
Step 1: Consider the consecutive nonscattering elements as being one nonscattering element and determine its directional front and back reflectances as well as its upward and downward transmittances,

Step 2: Derive its effective reflectances and transmittances in accordance with the angular distribution of incoming lights at the front and back sides,

Step 3: Define its transfer matrix from these effective reflectances and transmittances. This transfer matrix can be multiplied with the ones attached to diffusing layers.

Since many colored objects are composed of diffusing and nonscattering layers, application of the model may be illustrated from plenty of examples. We propose to expound the model through a simple case: a paper covered by a nonscattering film. The Williams-Clapper model, whose extension to halftones has been presented in Chapter 2, can be considered as the special case where the film is pasted to the paper. Then, we will study the color of papers covered with the stacks of blue acetate films studied in the previous section. Finally, we will consider a paper covered by nonscattering layers on its two sides. This latter model will be the first step towards the transmittance model for duplex halftone prints which will be the subject of the next section.

11.2. Paper covered with an acetate film

Let us consider a matte paper sheet whose spectral reflectance factor $\rho_\lambda$ and spectral transmittance factor $\tau_\lambda$ are measured in respect to a perfect diffuser with a diffuse-eight degree geometry. It is assumed to be a Lambertian reflector: as it receives the Lambertian flux $I_\lambda$, it reflects a Lambertian flux $r I_\lambda$. The detector captures the radiance $L_{r\lambda} = I_{r\lambda}/\pi$ at 8° to the normal. When the paper is replaced by a perfect diffuser, the detector captures the radiance $L_{ref\lambda} = I_\lambda/\pi$. The ratio $L_{r\lambda}/L_{ref\lambda}$ corresponds to the reflectance factor of the paper, which also coincides with the ratio of irradiances $I_{r\lambda}/I_\lambda$, therefore with the reflectance of the paper (rigorously, it is a bi-hemispherical reflectance which accounts for all incoming and outgoing lights over the hemisphere, see Chapter 1). The same considerations apply to the transmittance where incident light comes from the opposite side of the paper.

On top of the paper, we place a nonscattering colored film without optical contact. The film's spectral transmittance $T_\lambda$ is measured at normal incidence. Assuming that its refractive index $n_1$ is known, we can deduced the film’s normal transmittance $T_\lambda$ according to by Eq. (65).

By covering the paper with the film, we obtain a multilayer composed of diffusing and nonscattering elements. The multiple reflection process between them is shown in Fig. 34.
We want to predict the specimen's reflectance in respect to the diffuse-eight degree geometry. Let us follow the procedure in three steps leading to its transfer matrix.

*Step 1:* we want the film's directional reflectance $R_\lambda(\theta)$ and transmittance $T_\lambda(\theta)$. They are given by equations (61) and (194).

*Step 2:* we want the film's effective reflectances and transmittances in respect to the illumination and observation geometry. Since the film receives Lambertian light on its front side, the downward transmittance $T_{in}$ is the Lambertian transmittance derived from the directional transmittance $T_\lambda(\theta)$ according to the formula (182):

$$T_{in} = \int_0^{\pi/2} T_\lambda(\theta) \sin 2\theta d\theta$$

The front reflectance $r_s$ corresponds to the fraction of incident flux which is reflected by the film and perceived by the detector. Since only the radiance being in the regular direction in respect to the detector has a chance to be observed (it corresponds to a fraction $1/\pi$ of the incident flux), $r_s$ is given by

$$r_s = \frac{1}{\pi} R_\lambda(8^\circ)$$

where $R_\lambda$ is the film's directional reflectance given by Eq. (61).

On the back side, the film also receives a Lambertian flux from the paper. Its reflectance $r_i$ is

$$r_i = \int_0^{\pi/2} R_\lambda(\theta) \sin 2\theta d\theta$$

The upward transmittance $T_{out}$ corresponds to the fraction $1/\pi$ of Lambertian flux coming from the paper which is perceived by the detector at $8^\circ$:

$$T_{out} = T_\lambda(8^\circ) / \pi$$
Step 3: With the consider illumination and observation geometry, the transfer matrix of the film is

\[ S = \frac{1}{T_{in}} \begin{pmatrix} 1 & -r_j \\ r_s & T_{in}T_{out} - r_s r_j \end{pmatrix} \] (203)

The transfer matrix attached to the paper is defined from its front reflectance \( \rho_\lambda \), its upward transmittance is \( \tau_\lambda \), its downward transmittance equal to \( \tau_\lambda \) according to the nonpolarity principle, and its back reflectance \( \rho_\lambda' \) (which is not specified but we will see that we have no need of it):

\[ M = \frac{1}{\tau_\lambda} \begin{pmatrix} 1 & -\rho_\lambda' \\ \rho_\lambda & \tau_\lambda - \rho_\lambda \rho_\lambda' \end{pmatrix} \] (204)

The transfer matrices are multiplied by respecting the front-to-back staking order of the two components. From the transfer matrix \( SM \), by referring to equations (147), we deduce the front reflectance of the paper covered by the film:

\[ R_\lambda'' = r_s + \frac{T_{in}T_{out}\rho_\lambda}{1 - \rho_\lambda} \] (205)

and its back transmittance:

\[ T_\lambda'' = \frac{\tau_\lambda T_{out}}{1 - \rho_\lambda} \] (206)

By dividing the stack's reflectance expression (205) by the reflectance \( 1/\pi \) measured on a perfect diffuser, we obtain the reflectance factor which would be measured with a commercial spectrophotometer (see Chapter 1, Sections 3.8 and 3.10). Likewise, the transmittance expression (206) divided by the transmittance \( 1/\pi \) measured without any sample (air is the perfect transmitter) gives the transmittance factor. The reflectance and transmittance factors have therefore similar expressions as equations (205) and (206) where the terms \( 1/\pi \) in \( r_s \) and \( T_{out} \) are removed. We remark that the transmittance predictions require measuring both reflectance and transmittance of the paper alone, whereas for reflectance predictions, only the reflectance measurement is needed.

In order to illustrate experimentally this model, we used twelve papers printed by inkjet on their front side. As printed colors, we took a selection of fulltone colors (cyan, magenta, red, green, blue and black) as well as halftone colors (cyan and magenta halftones). The reflectance and transmittance of these papers were measured with a di:8° geometry. Then, we deposited on them a blue acetate film whose normal transmittance spectrum is plotted in Fig. 35. Fig. 35 shows the modification of reflectance spectrum due to the presence of the film on top of the magenta sample (paper coverage at 100% by magenta ink). The reflectance and transmittance were measured again and predicted according to equations (205) and (206).
respectively. The differences between predictions and measurement are expressed in terms of color difference by computing the CIELAB $\Delta E_{94}$ value in respect to the D65 standard illuminant. In reflectance mode, we obtained an average $\Delta E_{94}$ value of 0.46 for the 12 samples, which proves the excellent accuracy of the model. In transmittance mode, satisfying predictions are also obtained, even though slightly less accurate that in reflectance model with an average $\Delta E_{94}$ value of 1.15.

In a second experiment, we used 125 printed CMY halftones printed by the Canon Pixma Pro9500 inkjet printer on the Canon MP101 paper, and covered them by a standard transparent film for electrophotographic printing. Predictions were performed in reflectance mode and compared to the measured spectra. The same excellent prediction accuracy is proved by the obtained average $\Delta E_{94}$ value of 0.50. This second experiment is representative of the model's capacity to predict the changing of spectral reflectance of prints when they are put under glass or under a protection film. This changing of reflectance, thereby of color, is not only due to the wavelength-dependent absorption of light by the protection film, but also on the direct reflection of light by the plate and the multiple reflections of light between the film and the print.

Fig. 35. Spectral reflectance of (solid line:) a magenta printed paper and of (dashed line:35) this paper covered by a blue acetate film of normal transmittance $t_\lambda$.

11.3. Paper coated by a gelatin layer (Williams-Clapper model)

In 1953, Williams and Clapper proposed a prediction model for the spectral reflectance of glossy photographs, which are made of a paper support coated with a colored gelatin layer [91]. The paper substrate is diffusing whereas the gelatin is nonscattering. This configuration is closed to the one studied above except that the paper and the gelatin are in optical contact, i.e. there is no optical interface between them.
The Williams-Clapper reflectance formula is usually derived from geometrical series; we propose derive it from transfer matrices. We have three elements, from front to back: 1) the surface, assumed to be flat, 2) the gelatin layer of normal transmittance \( t_{\lambda} \) and refractive index \( n_1 \), and 3) the paper substrate, assumed to be Lambertian, of front reflectance \( \rho_\lambda \). The two first elements are nonscattering. The directionality principle therefore imposes to consider them as a single element, which will be called the interfaced layer. Its directional transfer matrix, denoted as \( S(\theta_0) \), is the product of the interface’s transfer matrix given by Eq. (168) and the gelatin’s transfer matrix given by Eq. (172):

\[
S(\theta_0) = \frac{1}{T_{01}(\theta_0) t_{\lambda}^{2/\cos \theta_1}} \left( \frac{1}{R_{01}(\theta_0)} \left[ T_{01}(\theta_0) T_{10}(\theta_1) - R_{01}(\theta_0) R_{10}(\theta_1) \right] t_{\lambda}^{2/\cos \theta_1} \right)
\]

(207)

where \( \theta_0 \) denotes the orientation of light in air, \( \theta_1 = \arcsin(\sin \theta_0 / n_1) \) the corresponding orientation in the gelatin and \( R_{01}(\theta_0) \) the Fresnel reflectivity of the interface. From the expression of \( S(\theta_0) \), we deduce that the front and back reflectances are respectively \( R_{01}(\theta_0) \) and \( R_{10}(\theta_1) \). The downward and upward transmittances, which are equal according to the transmittance nonpolarity principle, are \( T_{01}(\theta_0) t_{\lambda}^{1/\cos \theta_1} \). The exponents can be expressed as functions of \( \theta_0 \) thanks to the following transformation:

\[
\cos \theta_1 = \cos(\arcsin(\sin \theta_0 / n_1)) = \left[ 1 - (\sin \theta_0 / n_1)^2 \right]^{1/2}
\]

(208)

Now that the directional reflectances and transmittances of the nonscattering multilayer are known, let us calculate its effective ones in regard to the measuring geometry, e.g. the diffuse-8° geometry as in the previous example of the paper covered with a film. The front reflectance \( r_s \), the downward transmittance \( T_{in} \) and the back reflectance \( r_i \) have similar expressions as the ones given by equations (200), (199) and (201) respectively, by considering the directional functions derived above

\[
\begin{align*}
r_s &= \frac{1}{\pi} R_{01}(8^\circ) ; & r_i &= \int_0^{\pi/2} R_{10}(\theta_1) t_{\lambda}^{2/\cos \theta_1} \sin 2\theta_1 d\theta_1 \\
T_{in} &= \int_0^{\pi/2} T_{01}(\theta_0) \left[ 1 - (\sin \theta_0 / n_1)^2 \right]^{-1/2} \sin 2\theta_0 d\theta_0
\end{align*}
\]

(209)

The two integrals above can be accurately replaced by analytical functions:

\[
T_{in} = t_{01} \mu \quad \text{and} \quad r_i = r_0 \frac{\exp(t_{i}^\gamma) - 1}{\exp(1) - 1}
\]

(210)

where \( t_{01} \) is the Lambertian transmittance of the surface at the air-side, \( r_0 \) is the Lambertian reflectance of the surface at the gelatin-side (see Chapter 1, Section 4.6) and \( \gamma \) and \( \mu \) are coefficients minimizing the difference between the integrals and the approximation functions for a given refractive index.
Regarding the upward transmittance $T_{out}$, let us recall that it describes the transmission of radiance from the Lambertian support to the detector. Its general expression is given by Eq. (181). When the front and back surrounding media have different refractive indices, which is the case here, the transmittance accounts for the changing of solid angle due to the refraction: a squared ratio of their refractive indices appears [see Eq. (181)]. Since the index of air is 1 the refractive index of the medium where the Lambertian light is emitted is $n_1$, we have:

$$T_{out} = \frac{T_{10}(8^\circ)}{\pi n_1^2}$$

(211)

We can now describe the multiple reflection process of light between the interfaced layer and the paper substrate. It is similar to the one featured by Fig. 34 and the transfer matrices $M$ and $S$ attached respectively to the paper and the interfaced layer are identical to the ones expanded in equations (204) and (203), except that $r_s$ is now given by (209), $T_{in}$ and $r_i$ by (210) and $T_{out}$ by (211). We obtain the following reflectance expression, where the term into square brackets coincides with the reflector factor defined in respect to the perfect diffuser:

$$R_\lambda^* = \frac{1}{\pi} \left[ R_{01}(8^\circ) + \frac{1}{n_1^2} \cdot \frac{T_{10}(8^\circ) t_{01} \rho_\lambda}{1 - r_{01} \left( \frac{\exp[-1]}{e-1} \right) \rho_\lambda} \right]$$

(212)

In their original contribution, Williams and Clapper considered the $45^\circ:0^\circ$ measuring geometry. Their reflectance formula was slightly different, since in their case the interfaced layer has a front reflectance zero and a downward transmittance related to directional incident light oriented at $45^\circ$. The formula (212) obtained for a diffuse-directional geometry was first derived by Shore and Spoonhower [92].

Note that the reflectance of the paper substrate cannot be measured directly due to the presence of the paper-air interface. In order to deduce it from measurement, we can take the paper coated with gelatin in an area where the gelatin is uncolored, i.e. it normal transmittance is 1. If available, we can equivalently take the paper without gelatin coating. In this case, the reflectance $R_\lambda^*$ given by (212) becomes

$$R_\lambda^* = \frac{1}{\pi} \left[ R_{01}(8^\circ) + \frac{1}{n_1^2} \cdot \frac{T_{10}(8^\circ) t_{01} \rho_\lambda}{1 - r_{01} \rho_\lambda} \right]$$

(213)

By assuming that $n_1$ is known, we can compute the terms $R_{01}(8^\circ)$, $T_{10}(8^\circ)$, $r_{01}$ and $t_{01}$, and only background’s reflectance $\rho_\lambda$ remains unknown. We obtain it by reversing Eq. (213):

$$\rho_\lambda = \left[ r_{01} + \frac{T_{10}(8^\circ) t_{01}}{n_1^2 \left[ \pi R_\lambda^* - R_{01}(8^\circ) \right]} \right]^{-1}$$

(214)

If the measuring device provides a reflectance factor, the factor $\pi$ in (214) must be removed.
11.4. Stack of films on top of a diffusing background

The matrix model proposed for paper covered by film and for paper coated by a gelatin can be extended in a straightforward manner to any diffusing support topped by nonscattering multilayer. Another example is the stacks of acetate films studied in the previous section deposited on a diffusing background with flat surface. The prediction of their spectral reflectance allows studying the color variation as a function of the number of films, the color of the background and the refractive indices.

The acetate has an optical index $n_1$. The interstices between the acetate films are filled by a clear fluid which may be air or a liquid (index $n_2$). The diffusing background has an index $n_3$. The interstice between the background and the last film is filled with the same medium as between the other films. In this multilayer, the nonscattering component is composed of an alternation acetate layers and slices of a medium 2, bounded by the air-acetate interface at the front side, acetate-fluid interfaces in the stack, and the fluid-background interface at the back side (Fig. 36). In order to determine the transfer matrix of this nonscattering component, we follow the procedure in three steps recommended in Section 4.1.

First, we want to determine its directional reflectances and transmittances. We denote as $\theta_0$ the incident angle in air, $\theta_i = \arcsin(\sin \theta_0 / n_i)$ the subsequent angles of light in the medium of refractive index $n_i$, $F_j(\theta_i)$ the transfer matrices of the interfaces between media of refractive indices $n_i$ (on the front side) and $n_j$ (on the back side), and $L_i(\theta_i)$ the transfer matrix of each acetate layer. For $k$ films, the transfer matrix $S(\theta_0)$ of the nonscattering multilayer is given by
\[
S(\theta_0) = F_{01}(\theta_0) L_1(\theta_1) F_{12}(\theta_1) [F_{21}(\theta_2) L_4(\theta_1) F_{12}(\theta_1)]^{k-1} F_{23}(\theta_2)
\]  

(215)

The directional reflectances and transmittances issued from \(S(\theta_0)\) are too long to be written here. Let us simply denote the front reflectance as \(R_S(\theta_0)\), the back reflectance as \(R'_S(\theta_3)\), the downward transmittance as \(T'_S(\theta_3)\) and the upward transmittance \(T'_S(\theta_0)\).

Secondly, we want to determine the effective reflectances and transmittance of the nonscattering multilayer, by considering the illumination and observation geometries. Since the same geometries are used as in the two previous section, we have similar expressions for \(r_s\), \(T_{in}\), \(r_i\) and \(T_{in}\) as in equations (199) to (202), adapted to the directional reflectances and transmittances obtained above:

\[
\begin{align*}
    r_s &= \frac{1}{\pi} R_S(8^\circ) \\
    T_{in} &= \int_0^{\pi/2} T_S(\theta_0) \sin 2\theta_0 d\theta_0 \\
    r_i &= \int_0^{\pi/2} R'_S(\theta_3) \sin 2\theta_3 d\theta_3 \\
    T_{out} &= \frac{T'_S(8^\circ)}{\pi n_3^2}
\end{align*}
\]  

(216)

In the third step, we establish the effective transfer matrix of the nonscattering multilayer, which is similar to the one given in Eq. (203). Once multiplied with the transfer matrix of the background, similar to the one of Eq. (206) even though the background transmittance is not specified (we have no need of it since we only want to predict the reflectance of the specimen; we can therefore give it any value), we obtain the following reflectance expression, similar to the one given in Eq. (205):

\[
R'_s = r_s + \frac{T_{in} T_{out} \rho_\lambda}{1 - \eta \rho_\lambda}
\]  

(217)

The interest of this model lies on its analytical expression which is a function of physical parameters. Let us recall which these parameters are: the normal transmittance \(t_\lambda\) of the acetate layers, the Lambertian reflectance \(\rho_\lambda\) of the substrate, the refractive indices of the acetate \((n_1)\), of the medium fulfilling the interstices \((n_2)\) and of the substrate \((n_3)\), and the number \(k\) of films.

In order to verify the accuracy of the model, we considered the same blue acetate film as previously as well as different types of diffusing backgrounds: green glossy paper and white glossy PVC. One to five acetate film(s) were deposited on them with air interstices. The reflectance spectra of these specimens were measured with a diffuse-8° geometry. They are plotted in Fig. 37, as well as the corresponding predicted spectra.
Fig. 37. Measured (solid lines) and predicted (dashed lines) spectral reflectances of 1 to 5 acetate films on top of (a) white PVC and (b) green paper with respective reflectance $R_p$ plotted in dotted line. $R_\infty$ denotes the reflectance of an infinite stack of films.

To better understand the evolution of the specimen reflectance when incrementing the number of films, we subdivide the stack reflectance expression (217) into a first reflectance component independent of the background, represented by the term $r_s$, and a second reflectance component involving the support’s intrinsic reflectance, represented by the fraction. The first component corresponds approximately to the reflectance of the stack without background and thus follows the evolution described in the previous section: when incrementing the number of films, it increases if the interstices are filled by air or liquid and decreases if the liquid has the same refractive index as the acetate. The second component includes the downward transmission across the stack (term $T_{in}$) and the upward transmission from background to air (term $T_{ex}$). Therefore, it decreases in a similar manner as the stack transmittance when a film is added. The balance between first and second components is determined by the support reflectance. In the case of a highly reflecting support such as the white PVC support, the second component is dominant independently of the interstice medium. Incrementing the number of nonscattering films decreases the specimen reflectance, therefore the lightness. The chroma reaches a maximum. This is well-known in the case of oil paintings, particularly regarding the art-glaze technique where a white diffusing board is coated with several weakly pigmented layers [93,94]. The number of films yielding the highest chroma is approximately half the number of films for which the chroma of a pile alone is maximal in transmission mode. This can be explained by the fact that light reflected by the support traverses twice the pile. When the number of films tends to infinity, the support has no influence any more: the pile reflectance tends to the same reflectance $R_\infty$ with or without background. When the background is colored, the balance between the first and the second components may be different for each wavelength. This appears clearly in the case of the green paper (Fig. 37), where adding a film increases the reflectance in the short
wavelength domain and decreases the reflectance beyond 500 nm. At the wavelength \( \lambda = 500 \) nm, all the specimens have the same reflectance, included the support alone. We observe that this invariance phenomenon occurs precisely at the wavelength where the support reflectance \( R_g \) coincides with the infinite air-bound pile reflectance \( R_\infty \), which remains constant when adding films. This is well-known in the case of oil paintings, particularly regarding the art-glace technique where a white diffusing board is coated with several weakly pigmented layers [95].

12. SPECTRAL PREDICTION MODEL FOR HALFTONE PRINTS

A halftone is a mosaic of colored areas resulting from partial overlap of dot screens of primary inks. The areas with no ink, those with a single ink layer, and those with two or three superposed ink layers are each considered as a distinct colorant, also called Neugebauer primary (Fig. 38)

For three primary inks (e.g. cyan, magenta and yellow), one obtains a set of eight colorants: white (no ink), cyan alone, magenta alone, yellow alone, red (magenta & yellow), green (cyan & yellow), blue (cyan & magenta) and black (cyan & magenta & yellow). In classical clustered-dot or error diffusion prints, the fractional area occupied by each colorant can be deduced from the surface coverages of the primary inks according to Demichel’s equations [96]. These equations are valid in all cases where the ink halftone dots are laid out independently, e.g. in stochastic screening, in error diffusion, or in mutually rotated clustered dot screens. For cyan, magenta and yellow primary inks with respective surface coverages \( c \), \( m \), and \( y \), the surface coverages \( a_\alpha \) of the eight colorants are respectively:

\[
\begin{align*}
  a_w &= (1-c)(1-m)(1-y) \\
  a_c &= c(1-m)(1-y) \\
  a_m &= (1-c)m(1-y) \\
  a_y &= (1-c)(1-m)y \\
  a_{m+y} &= (1-c)my \\
  a_{c+y} &= c(1-m)y \\
  a_{c+m} &= cm(1-y) \\
  a_{c+m+y} &= cmy
\end{align*}
\]
The color rendering of printed surfaces may be assessed by printing all possible halftones on the considered paper, inks, halftoning technique and printing system, and measuring all these colors. However, this handmade procedure is cumbersome to perform and needs to be extensively repeated each time either the paper, the inks, the halftoning technique or the printing system is changed. There is therefore a strong interest in predicting all these colors using a model calibrated from a small number of printed patches. Every prediction model exhibits a general equation involving spectral parameters describing the absorption by the inks and the reflection by the paper, as well as the respective surface coverages by the inks.

We propose here to review classical spectral prediction models developed for typical halftoning techniques, such as periodical or random clustered dot centering halftones.

### 12.1. The Spectral Neugebauer model

Let us assume, as a first approximation, that the reflectance of the halftone print is the sum of the colorant-on-paper reflectances \( R_k(\lambda) \) weighted by their respective surface coverages \( a_k \). We obtain the spectral Neugebauer reflectance model [97]

\[
R(\lambda) = \sum_{k=1}^{8} a_k R_k(\lambda)
\]  

(219)

Fig. 39 shows an example of prediction performed with the Neugabauer model in the case of a cyan ink halftone at nominal surface coverage 0.5. The halftone contains two colorants: paper white and cyan. Their reflectance spectra are plotted in black and cyan solid lines in the figure. As each colorant occupies half the surface, the spectral reflectance of the halftone should be the average between the two colorant spectra. However, we see that the predicted spectrum (cyan dotted line) is far from the measured one (light cyan solid line).

![Fig. 39. Spectral reflectance of paper, cyan 100% and cyan 50%. Measured (solid lines) and predicted by the Neugabuer model with a surface coverage for the cyan ink of 0.5 (dotted) or 0.59 (dashed line).](image)

We may think that at the printing time the cyan ink spreads on the surface and therefore occupies as larger area on the surface. By fitting the surface coverage of cyan ink so as to
minimize the difference between predicted and measured spectra, we find an optimal surface coverage of 0.59, which not physically well plausible. The spectrum predicted with this surface coverage (cyan dashed line) is much closer to the predicted spectrum as before, but its deviation from the measured spectrum is still important: the equivalent color distance would be well perceptible (CIELAB $\Delta E_{94} = 2.2$).

12.2. The Yule-Nielsen modified Spectral Neugebauer model

The linear Eq. (219) does not predict correctly the reflectance of halftones printed on paper due to the scattering of light within the paper bulk and the multiple reflections between the paper bulk and the print-air interface, which induce lateral propagation of light from one colorant area to another. This phenomenon is known as the Yule-Nielsen effect [98, 99]. In order to account for this effect, we follow the same approach as Viggiano [100] by raising all the transmittances in Eq. (219) to a power of $1/n$. We obtain the Yule-Nielsen modified spectral Neugebauer (Yule-Nielsen modified Spectral Neugebauer, or simply Yule-Nielsen) equation for the transmittance of a color halftone printed on the recto side:

$$R(\lambda) = \left[ \sum_{k=1}^{8} a_k R_k^{1/n} (\lambda) \right]^n \quad (220)$$

A second phenomenon well-known in halftone printing is the spreading of the inks on the paper and on the other inks. In order to obtain accurate spectral transmittance predictions, effective surface coverages need to be known. They are deduced from the measured transmittance spectra of a selection of printed halftones, called calibration patches.

![Fig. 40. Measured reflectance spectra of paper, cyan 100% and cyan 50% (solid lines) and predicted spectra with the Yule-Nielsen model with a surface coverage for the cyan ink of 0.5 (dotted line) and 0.59 (dashed line).](image)

12.3. Experimental analysis of the Yule-Nielsen equation

Let us test the Yule-Nielsen model with the previous example of the cyan halftone. The matching between the predicted and measured spectra, plotted in Fig. 40, is considerably
improved (CIELAB $\Delta E_{94} = 0.24$). The prediction was performed with $n = 4$ and with an optimal surface coverage $a_c = 0.52$. The surface coverage represents the effective surface coverage of the cyan ink dots according to the Yule-Nielsen model.

In order to better understand the capacity of the Yule-Nielsen model to fit the actual reflectances of a halftone, we establish a relationship between the attenuation of light due to the halftone ink, no attenuation by the support and maximal attenuation by the corresponding solid ink. For this purpose, by denoting as $R_0$, $R_a$ and $R_1$ the spectral reflectances of patches where the ink has the respective surface coverages of 0, $a$ and 1, we propose to convert the Yule-Nielsen equation written for single ink halftones

\[
R_a(\lambda) = [ (1-a) R_0^{1/n}(\lambda) + a R_1^{1/n}(\lambda) ]^n
\]

(221)

into the following equation obtained by dividing Eq. (221) by $R_0$

\[
\frac{R_a}{R_0} = [ 1 - a + a (R_1/R_0)^{1/n} ]^n
\]

(222)

This relation of the form $R_a/R_0 = f(R_1/R_0)$ can be verified experimentally from the measurements of $R_0$, $R_a$ and $R_1$ which provide as many points as values contained in the measured spectra. This relation also enables observing, for the fitted values of $a$ and $n$, the variation of $R_a/R_0$ as a function of $R_1/R_0$ from 0 (reflectance $R_1$ is zero, which means that all light is absorbed by the ink) to 1 ($R_1$ is equal to $R_0$, which means that the ink is transparent).

Fig. 41 shows two examples of halftones of cyan ink printed at 0.5 nominal surface coverage and measured with diffuse-8° geometry with specular reflection included. On the right of the figure, the measured spectra of $R_0$, $R_a$ and $R_1$ are plotted as well as the predicted spectrum $R_a$ for $n = 10$ and for the $a$ value fitted so as to minimize the deviation from the measured spectrum. On the left of the figure, the theoretical curve $R_a/R_0 = f(R_1/R_0)$ is plotted in solid black line for the fitted value of $a$. The straight line shows how it would vary with the same value of $a$ and $n = 1$ (Spectral Neugebauer model). The black squares correspond to the points $(R_1/R_0; R_a/R_0)$ derived from measurements.

In the first example, we observe an excellent agreement between prediction and measurement. In the second example, the difference between prediction and measurement is substantial with a $\Delta E_{94}$ value of 1.25. Such a medium prediction accuracy often occurs when the ink is very absorbing in a certain wavelength domain. In this wavelength domain, the ink layer prevents light from entering the paper and its surface exhibits a colored sheen whose reflection spectrum is very different from the transmittance of the ink.
1) Inkjet Cyan ink at 0.5 nominal surface coverage on Canon MP101 mat paper printed with the Canon Pixma Pro9500 inkjet printer:

2) Cyan ink at 0.5 nominal surface coverage on Canon PP201 glossy paper printed with the Canon Pixma Pro9500 mark II inkjet printer:

Fig. 41: Comparison of predicted and measured spectral reflectances of inkjet cyan ink printed at 0.5 nominal surface coverage. Left: diagrams representing the curve predicted according to Eq. (220) (black solid curve) and the measured points (\( R_1/R_0 \); \( R_a/R_0 \)) at the different wavelengths; Right: Spectral reflectances of the unprinted paper (measured \( R_0(\lambda) \)), of the halftone (measured and predicted \( R_a(\lambda) \)) and of the solid ink patch (measured \( R_1(\lambda) \)).

The example presented in Fig. 41-b illustrates this particularity: the cyan ink layer is highly absorbing between 560 nm and 730 nm. In this wavelength domain, the reflectance of the solid ink patch is mainly due to the specular component, i.e. a sheen whose reddish appearance is explained by the fact that the reflectance increases as the wavelength approaches 730 nm. This sheen is less pronounced in the halftone ink, because the light scattered in the paper bulk transits from inked and non-inked areas. As a consequence, the non-linear interpolation defined by the Yule-Nielsen equation between the spectra reflectances \( R_0 \) and \( R_1 \) is not able to predict correctly the spectral reflectance of the halftone
ink, especially in the spectral domain where the ink is more absorbing. When the specular component is discarded from the measurements, the sheen effect does not appear and the prediction accuracy is improved. The optimal $n$-value is generally lower, which coincides with the improved agreement between predicted and measured spectra.

12.4. Calibration of the model

The Spectral Neugebauer and Yule-Nielsen models are calibrated from a reduced number of printed patches, displayed in Fig. 42.

First of all, the spectral reflectances of the eight colorants are measured using a spectrophotometer (based on a given measuring geometry, e.g. the 45°:0° geometry). The corresponding patches are shown in row A of Fig. 42. In a given halftone, the surface coverage of each colorant should be easily determined using the Demichel equations (218). However, since the inks may spread at the paper surface, their effective surface coverage is a priori not known and must be estimated from spectral reflectance measured on real printed halftones.

**Ink spreading assessment methods**

When considering a given printing setup, one is a priori unable to estimate how much the different wavelengths are reflected by the paper or absorbed by the inks and how much the ink dots spread on the paper surface, until we perform specific spectral measurements. The spectral properties of the paper and inks are represented in each model by spectral
reflectances or transmittances deduced from the spectral reflectance of full-tone colors. The growth of the ink dots, i.e. the dot gain, is assessed by establishing the correspondence between nominal and effective surface coverages for each ink, thanks to nominal-to-effective surface coverage functions, also called ink spreading functions. The spectral reflectances, respectively transmittances, and the ink spreading functions are computed in a calibration procedure which ensures the ability of the model to account for the specific properties of the considered print.

Once the spectral parameters are obtained, we can start assessing the dot gain. The correspondence between the effective surface coverage of the inks and their nominal ones is represented by ink spreading functions as featured in Fig. 43. Although each model (except the SN model) already accounts for optical dot gain, the effective ink surface coverages may also compensate for a possible under- or overestimation of optical dot gain. Two dot gain assessment methods are possible: the basic ink spreading (BIS) method where the spreading of each ink is assessed independently of each other ink, and the superposition-dependent ink spreading (SDIS) method where ink superposition configurations are taken into account.

**Basic ink spreading (BIS) method**

Each ink $i$ is printed alone on paper at the nominal surface coverages $q_i = 0.25, 0.5$ and $0.75$, which corresponds to the 9 color patches represented in rows B,C and D of Fig. 42. Let us denote as $R_i^{\text{meas}}(q, \lambda)$ their respective spectral reflectance. These halftones contain two colorants: the ink which should occupy a fractional area $q_i$ and the paper white which should occupy the fractional area $1 - q_i$. Applying the model’s equation with these two colorants and these surface coverages should predict a spectral reflectance $R_i^{\text{pred}}(q, \lambda)$ equal to the measured one. However, due to the fact that the effective ink surface coverage is different from the nominal one, these two reflectances are not the same. We thus fit the effective surface coverage $q'_i$ as the $q_i$ which minimizes the deviation between predicted and measured spectra, by quantifying the deviation either by the sum of square differences of the components of the two spectra, i.e.

$$q'_i = \arg \min_{0 \leq q \leq 1} \sum_{\lambda = 380 \text{nm}}^{730 \text{nm}} \left[ R_i^{\text{pred}}(q, \lambda) - R_i^{\text{meas}}(q, \lambda) \right]^2$$  \hspace{1cm} (223)

or by the sum of square difference of the components of their logarithm, i.e.

$$q'_i = \arg \min_{0 \leq q \leq 1} \sum_{\lambda = 380 \text{nm}}^{730 \text{nm}} \left[ \log R_i^{\text{pred}}(q, \lambda) - \log R_i^{\text{meas}}(q, \lambda) \right]^2$$  \hspace{1cm} (224)

or by the corresponding color difference given e.g. by the CIELAB $\Delta E_{94}$ metric

$$q'_i = \arg \min_{0 \leq q \leq 1} \Delta E_{94}\left(R_i^{\text{pred}}(q, \lambda), R_i^{\text{meas}}(q, \lambda)\right)$$  \hspace{1cm} (225)
Equation (223) is the most classical way of determining the effective surface coverage. Taking the log of the spectra as in Equation (224) has the advantage of providing a higher weight to lower reflectance values where the visual system is more sensitive to small spectral differences. Fitting $q'_i$ from the color difference metric sometimes improves the prediction accuracy of the model in terms of color differences but complicates the optimization. Even at the optimal surface coverage $q'_i$, the difference between the two spectra is rarely zero and provides a first indication of the prediction accuracy achievable by the model for the corresponding print setup.

Once the 9 effective surface coverages are computed, assuming that the effective surface coverage is 0, respectively 1, when the nominal surface coverage is 0 (no ink), respectively 1 (full coverage), we obtain three sets of $q'_i$ values which, by linear interpolation, yield the continuous ink spreading functions $f_i$ (Fig. 43).

As an alternative, one can print halftones at nominal surface coverage 0.5 only and perform parabolic interpolation. The number of patches needed for establishing the ink spreading curves is then reduced to 3 (row C in Fig. 42).

**Superposition-dependent ink spreading (SDIS) method**

As explained in Section 5, one often observes that the amount of ink spreading depends on whether the ink is alone on the support or superposed with other inks. In addition to the effective surface coverages computed from single-ink halftones alone on paper, the SDIS method includes effective surface coverages computed from the single-ink halftones superposed with a solid layer of either one or the two other inks. 36 color patches need to be printed, represented by the rows B to G in Fig. 42.

The effective surface coverages of the halftones are obtained in the same way as in the BIS method, by considering the appropriate two colorants for each halftone. We obtain 12 sets of effective surface coverages, providing 12 ink spreading curves $q' = f_{i/j}(q)$ where the
subscript $i/j$ denotes ink halftone $i$ superposed with solid colorant $j$ (see Fig. 44). By printing halftones at nominal surface coverage 0.5 only and performing parabolic interpolation, the number of patches to be printed and measured is thus reduced to 12 (rows C and F in Fig. 42).

![Fig. 44. Example of ink spreading curves $f_{ij}$ obtained by linear interpolation of the effective surface coverages $q'_{ij}$ which are deduced from measurement of patches with single-ink halftones (ink $i$) printed at nominal surface coverages 0.25, 0.5 and 0.75 and superposed with a solid layer of colorant $j$.](image)

### 12.5. Prediction

Once the spectral reflectances and/or transmittances as well as the ink spreading functions are acquired and/or computed, the model is calibrated. We can then predict the spectral reflectance of halftones with any nominal ink surface coverages $c$, $m$, and $y$. If the dot gain was calibrated using the BIS method, the ink spreading functions $f_i$ directly provide the effective surface coverages $c'$, $m'$, and $y'$ of the three inks:

$$c' = f_c(c); \quad m' = f_m(m); \quad y' = f_y(y)$$

These effective ink surface coverages are plugged into the Demichel equations, which provide the effective surface coverages of the eight colorants. The general equation of the model finally predicts of the reflectance spectrum of the considered halftone.

If the dot gain was calibrated using the SDIS method, the nominal ink surface coverages $c$, $m$, and $y$ are converted into effective ink surface coverages $c'$, $m'$ and $y'$ by accounting for the superposition-dependent ink spreading. The effective surface coverage of each ink is obtained
by a weighted average of the ink spreading curves. The weights are expressed by the surface coverages of the respective colorants on which the ink halftone is superposed. For example, the weight of the ink spreading curve \( f_c \) (cyan halftone over the white colorant) is proportional to the surface of the underlying white colorant, i.e. \((1-m)(1-y)\). In the case of three halftoned inks, effective surface coverages are obtained by performing a few iterations with equations (74):

\[
c'(1-m')(1-y') = f_c(c) + m'(1-y')f_{c/m}(c) + (1-m')y'f_{c/y}(c)
\]
\[
m' = (1-c')(1-y')f_m(m) + c'(1-y')f_{m/c}(m) + (1-c')y'f_{m/y}(m) + c'y'f_{m/c+y}(m)
\]
\[
y' = (1-c'')(1-m'')f_y(y) + c''(1-m'')f_{y/c}(y) + (1-c'')m'y''f_{y/m}(y) + c''m'y''f_{y/c+m}(y)
\]

For the first iteration, \(c' = c\), \(m' = m\) and \(y' = y\) are taken as initial values on the right side of the equations. The obtained values of \(c\), \(m\) and \(y\) are then inserted again into the right side of the equations, which yields new values of \(c', m', y'\) and so on, until the values of \(c', m', y'\) stabilize. The effective surface coverages of the colorants are calculated by plugging the obtained values for \(c', m', y'\) into the Demichel equations. The spectral reflectance of the considered halftone is finally provided by the general equation of the model.

![Graph showing average \(\Delta E_94\) values for different halftone sets and models](image)

**Fig. 45.** Average \(\Delta E_94\) values between measured and predicted spectral reflectances provided by three models for different sets of halftones by using the BIS (abscissa) or SDIS (ordinate) method.

Regarding the ink spreading assessment method, the superposition-dependant ink spreading method (SDIS) improves considerably the prediction accuracy of the models compared to the basic ink spreading method (BIS). This is illustrated by diagram of Fig. 45 where the
coordinates of each point represent the prediction accuracies provided by the BIS method (in abscissa) and SDIS method (in ordinate). As all points are always below the dotted line of slope 1, we conclude that the SDIS method offers improved prediction accuracy with even a spectacular improvement in many cases.

12.6. The Clapper-Yule model

The model proposed in 1953 by Clapper and Yule from Kodak (Rochester, USA) [101] is a phenomenological model permitting the prediction of colors printed on diffusing substrates. The model relies on a closed-form equation obtained by describing the multiple reflections of light between the substrate and the print-air interface through the inks. The intrinsic reflectance of the substrate, the transmittance of the colorant layers and their respective surface coverages, and the reflectances and transmittances of the surface, derived from Fresnel's formulae, are the parameters of the model. The model assumes that the lateral propagation distance of light within the substrate, due to scattering, is much larger than the halftone dot size. Most photons therefore cross different areas in the halftone before exiting the print. The reflections and transmissions of light at the surface are explicitly taken into account depending on the print's refractive index, as well as the considered illumination and measuring geometries.

The general Clapper-Yule equation

The Clapper-Yule equation derives from the description of multiple reflections of light between the substrate and the print-air interface, represented by the Fig. 46.

Fig. 46: Multiple reflections between the print surface and the diffusing substrate.

On the one side, the substrate considered without any interface with air has a spectral reflectance \( r_g(\lambda) \). On the other side, the air-print interface surface is responsible for reflections and refractions of light. Regarding the light coming from air, a fraction \( r_s \) is
reflected towards the detector (this fraction may be zero according to the geometry of illumination and detection), and a fraction $T_{in}$ is refracted into the print. Regarding the diffuse light coming from the substrate, a fraction $r_i$ is reflected by interface, and a fraction $T_{ex}$ exits the print towards the detector.

Between the substrate and the interfaces, the inks form an absorbing halftone layer composed of the eight colorants displayed in Fig. 38. The spectral transmittances of the colorants are denoted as $t_k(\lambda)$ ($k = 1, \ldots, 8$) and their respective surface coverages $a_i$ are determined by the surface coverage of the three inks using the Demichel equations (218). For light traveling a single pathway through the halftone layer, the transmittance of the layer is simply

$$\sum_{k=1}^{8} a_k t_k$$

Note that the diffuse light coming from the substrate and being internally reflected by the air-print interface travels a double pathway through the absorbing layer. The reflectance of the interface + halftone later is

$$r_i \sum_{k=1}^{8} a_k t_k^2$$

where $r_i$ has been defined as the reflectance of the interface itself. Note that the transmittance of the halftone layer is assumed independent of the orientation of the light flowing through it. This makes sense in the case of pigment inks in which light is slightly scattered, or in the case where the inks penetrate the paper and actually become scattering layers. These two configurations are the most frequent in printing applications. However, in specific cases where the halftone layer can be assumed non-scattering, e.g. in digital photo printing, it would be more rigorous to take explicitly the orientation of light in the inks. This is the purpose of the Williams-Clapper model extended to halftones [102] presented in the next Section.

Let us now describe the whole multiple reflection process taking place between the substrate, halftone layer and air-print interface. We denote as $I_0$ and $I_1$ the downward fluxes illuminating the surface, respectively the substrate, and as $J_0$ and $J_1$ the upward fluxes exiting the surface, respectively the substrate. From Fig. 46, we deduce the following relations:

$$J_0 = r_s I_0 + (T_{ex} \sum a_i t_i) J_1,$$

$$I_1 = (T_{in} \sum a_i t_i) I_0 + (r_i \sum a_i t_i^2) J_1,$$

$$J_1 = r_g I_1$$

from which we deduce the reflectance $R$ of the halftone print

$$R = \frac{J_0}{I_0} = r_s + \frac{T_{in} T_{ex} r_g (\sum a_i t_i)^2}{1 - r_i r_g \sum a_i t_i^2}$$ (227)
This equation is the general Clapper-Yule equation. In their original paper, Clapper and Yule considered halftones of one ink, with surface coverage $a$ and transmittance $t$:

$$R = r_s + \frac{T_{in}T_{ex}r_g(1-a+at)^2}{1-r_r r_g(1-a+at^2)}$$

(228)

If one colorant covers the whole surface ("solid colorant layer"), the equation becomes

$$R = r_s + \frac{T_{in}T_{ex}r_g t^2}{1-r_r r_g t^2}$$

(229)

and in the special case where no ink is printed (white colorant), it expresses the reflectance of the printing support:

$$R = r_s + \frac{T_{in}T_{ex}r_g}{1-r_r r_g}$$

(230)

**Fresnel terms and measuring geometries**

The terms $T_{in}$, $T_{ex}$, $r_s$ and $r_r$ are derived from the Fresnel formulae for unpolarized light (see Section 4.4). The Fresnel reflectance of an interface between media 1 and 2, with respective indices $n_1$ and $n_2$, is denoted as $R_{12}(\theta)$ when the light comes from medium 1 at the angle $\theta$.

The terms $T_{in}$ depends on the illumination geometry. It is typically, the incident light is directional and one has

$$T_{in} = 1 - R_{12}(\theta)$$

or it is perfectly diffuse, and one has

$$T_{in} = 1 - \int_{\theta=0}^{\pi/2} R_{12}(\theta) \sin 2\theta d\theta$$

$T_{ex}$ depends on the measuring geometry. When the print is observed in one direction $\theta'$, it is

$$T_{ex} = (n_1 / n_2)^2 T_{12}(\theta')$$

where the factor $(n_2/n_1)^2$ comes from the change of solid angle containing the exiting radiance due to the refraction [see Section 4.5, Eq. (45)]. When an integrating sphere captures all light reflected by the print, it is

$$T_{ex} = 1 - \int_{\theta=0}^{\pi/2} R_{21}(\theta) \sin 2\theta d\theta$$

The term $r_s$ corresponds to the surface reflection, i.e. the gloss. It depends on the illumination and observation geometries, by it is typically 0.05 when it is captured, or 0 when it is discarded from measurement.
The reflectance $r_i$ is the fraction of diffuse light emerging from the substrate which is reflected by the surface. It corresponds to the Lambertian reflectance of flat interfaces (see Section 4.6, Eq. (49))

$$r_i = \int_{\theta=0}^{\pi/2} R_{21}(\theta) \sin 2\theta d\theta$$

Let us consider for example a print with refractive index 1.5, illuminated by directional light at 45° and observed at 0° (so-called 45°:0° measuring geometry). We have:

$$T_{in} = 1 - R_{12}(45°) = 0.95, \quad T_{ex} = T_{12}(0°) / n^2 = \frac{1}{n^2} \left[ 1 - \frac{(1-n)^2}{(1+n)^2} \right] = 0.42, \quad r_s = 0, \quad \text{and} \quad r_i = 0.6.$$  

The Clapper-Yule equation thus becomes

$$R = \frac{0.4r_g \left( \sum a_i t_i \right)^2}{1 - 0.6r_g \left( \sum a_i t_i^2 \right)}$$

**Calibration of model and prediction.**

The Clapper-Yule model is calibrated from the same printed color patches as the Yule-Nielsen model, i.e. the 8 solid colorant patches (row A in Fig. 42) from which are deduced the spectral parameters related to the substrate and the inks, and the same halftone patches as those needed for the inks spreading assessment.

By measuring the spectral reflectance $R_w(\lambda)$ of the unprinted support expressed by Eq. (230), one deduces the intrinsic spectral reflectance of the substrate thanks to the following formula:

$$r_g(\lambda) = \frac{R_w(\lambda) - r_s}{T_{in}T_{ex} + r_i \left( R_w(\lambda) - r_s \right)}$$  \quad (231)

Then, by measuring the spectral reflectances $R_i(\lambda)$ of the solid colorant patches expressed by Eq. (229), one deduces the intrinsic spectral reflectance of the substrate thanks to the following formula:

$$t_i(\lambda) = \sqrt{\frac{R_i(\lambda) - r_s}{r_g(\lambda) \left[ T_{in}T_{ex} + r_i \left( R_i(\lambda) - r_s \right) \right]}}$$  \quad (232)

The ink spreading assessment follows the same methods as those detailed in Section 12.4, by considering the Clapper-Yule equation in place of the Yule-Nielsen equation. Likewise, the spectral reflectance prediction for a given halftone is performed as detailed in Section 12.5 for the Yule-Nielsen model.
Experimental testing of the Clapper-Yule model

In order to assess the prediction accuracy of the model, predicted and measured spectra may be compared on sets of printed colors. As comparison metric, one generally uses the CIELAB $\Delta E_{94}$, obtained by converting the predicted and measured spectra first into CIE-XYZ tristimulus values, calculated with a D65 illuminant and in respect to a $2^\circ$ standard observer, and then into CIELAB color coordinates using as white reference the spectral reflectance of the unprinted paper illuminated with the D65 illuminant.

Because it assumes that the lateral propagation of light is large compared to the halftone screen period, the Clapper-Yule model is theoretically restricted to halftones with high screen frequency. For example, the model tested on two sets of 729 CMY colors printed with the same offset press on the same paper but at different frequencies, respectively 76 and 152 lines per inch (lpi), provides better predictions for the highest frequency (average $\Delta E_{94}$ of 0.98 units) than for the lowest one (average $\Delta E_{94}$ of 1.26 units). Nevertheless, the experience shows that the model may also perform well for middle and low frequencies: for a set of 40 CMY colors printed in inkjet at 90 lpi on supercalendered paper, the model achieves a fairly good prediction accuracy, denoted by the average $\Delta E_{94}$ of 0.47 units. Note that the average $\Delta E_{94}$ is 0.70 units when the ink superposition conditions are not taken into account in the ink spreading assessment.

Despite the simplicity of its base equation, the Clapper-Yule model is one of the most accurate prediction models for halftone prints. Its main advantage compared to other models such as the Neugebauer model or the Yule-Nielsen corrected Neugebauer model, is the fact that physical parameters are attached to the different elements composing the print (inks, paper and surface). The Fresnel terms can be adapted to the considered measuring geometry, which is particularly interesting when predictions are made for a geometry different from the one used for calibration. The model also enables controlling ink thickness variations at printing time by comparing the colorant transmittances in various halftones, whose log is proportional to the ink thickness [103]. Recent improvements and extensions have been proposed which enable predicting both reflectance and transmittance of halftone prints with increased accuracy thanks to the possibility to take into account the orientation of light into the inks or the possibility to consider different ink spreading when one ink is superposed with each other ink.

12.7. The Williams-Clapper model extended to halftone prints

In the same year 1953, Williams and Clapper from Kodak published the model for gelatin photographs already introduced in Section 11.3 [91]. In contrast with halftone prints, the gelatin layer is a uniform nonscattering layer. The model takes explicitly the orientation of
light in it. It can therefore be qualified as an “orientational” model in opposition to the Clapper-Yule model which would thus be a “non-orientational” model. Except this difference, both models are based on a similar description of the multiples reflections of light between the substrate and the print-air interface, as featured by Fig. 46. They therefore lie on similar formula, except the spectral, angular reflectances and transmittances of the interface + layer.

**Uniform coloring layer**

In the Williams-Clapper model, the coloring layer is explicitly assumed to be transparent. Let us first consider a uniform coloring layer. According to Beer’s law, light is attenuated within the ink layer as a function of the length of the traversed path. The length of the traversed path is a function of its orientation. When light crosses the layer perpendicularly, it is attenuated by a factor $t$ called the *normal transmittance* of the layer. When it is incident at a nonzero angle $\theta$, it traverses a path of relative length $1/\cos \theta$ within the ink layer and is, according to Beer’s law, attenuated by a factor $t^{1/\cos \theta}$. Since neither the coloring layer or the ink-air interface scatter light, the travel of light through them, therefore their transmittance, are determined by the angular distribution of the incident light. When the incident light is collimated, the angles of reflection and transmission are determined according to Snell’s laws and the reflectance and transmittance are derived from Fresnel’s formulae and Beer’s law. They are presented in Fig. 47. Note that angles $\theta_0$ (in air) and $\theta_1$ (in the print of optical index $n$) are related according to Snell’s laws, i.e. $\sin \theta_0 = n \sin \theta_1$. Angle $\theta_1$ is therefore $\arcsin \left(\sin \frac{\theta_0}{n}\right)$ and

$$\cos \theta_1 = \sqrt{1 - \left(\frac{\sin \theta_0}{n}\right)^2}$$

When the incident light is diffuse, the reflectance and the transmittance of the colored interface are obtained by integrating these angular functions over the range of incidence.

![Fig. 47: Reflectance and transmittance of the ink-air interface and (uniform) ink layer according to the Williams-Clapper model.](image)

Let us consider that the incident light crossing the interface and layer is either collimated or diffused. For collimated light at angle $\theta_0$, the transmittance of the interface and layer is

$$T_{\lambda} = T_{\lambda,0}(\theta_0)^{1/\cos \theta_1}$$

where $T_{\lambda,0}$ is the transmittance at angle $\theta_0$ and $\theta_1$ is determined from Snell’s law.
For collimated incident light at 45° with \( n = 1.5 \), \( T_{in}(t,\lambda) = 0.95t^{1.13}(\lambda) \).

For Lambertian incident light, the transmittance is

\[
T_{in}(t,\lambda) = \int_{\theta_0}^{\pi/2} T_{01}(\theta_0) \left[ t(\lambda) \right]^{(1/4)[1-(\sin \theta_0/n)^2]} \sin 2\theta_0 d\theta_0 ,
\]
e.g. for \( n = 1.5 \), \( T_{in}(t,\lambda) = 0.90t^{1.13}(\lambda) \).

The incident light specularly reflected towards the detector is denoted as \( r_s \) and can be zero depending on the measuring geometry.

Let us now consider the Lambertian light emerging from the substrate. Part of it crosses the layer and the interface and exits into air. If an integrating sphere is used to collect all this exiting light, the transmittance incorporates all orientations of light [92]:

\[
T_{ex}(t,\lambda) = \frac{T_{01}(\theta)}{n^2} \left[ t(\lambda) \right]^{(1/4)[1-(\sin \theta/n)^2]} \sin 2\theta d\theta ,
\]
e.g., for \( n=1.5 \), \( T_{ex}(t,\lambda) = 0.40t^{1.13}(\lambda) \).

If the detector captures a radiance in one direction at angle \( \theta \) from the normal (as a human observe would actually do with his eyes), the transmittance is

\[
T_{ex}(t,\lambda) = \frac{T_{01}(\theta)}{n^2} \left[ t(\lambda) \right]^{(1/4)[1-(\sin \theta/n)^2]} \sin 2\theta d\theta ,
\]
Part of the Lambertian light emerging from the substrate is internal reflected by the interface, while crossing twice the layer. The corresponding reflectance, which incorporates all orientations of the hemisphere, is

\[
r_i(t,\lambda) = \int_{\theta_0}^{\pi/2} \left[ t(\lambda) \right]^{(2/\cos \theta)} R_{10}(\theta) \sin 2\theta d\theta ,
\]
Finally, the Clapper-Yule equation for a substrate with own reflectance \( r_g(\lambda) \) coated with a uniform layer with normal transmittance \( t(\lambda) \) is

\[
R(\lambda) = r_s + T_{in}(t,\lambda) T_{ex}(t,\lambda) \frac{r_g(\lambda)}{1 - r_i(t,\lambda) r_g(\lambda)}
\]
For example, if the measuring geometry is based on collimated illumination at angle \( \theta \) and a detector at angle \( \theta' \), the reflectance is

\[
R(\lambda) = \frac{T_{01}(\theta) T_{01}(\theta') \left[ t(\lambda) \right]^{(1/4)[1-(\sin \theta/n)^2]+(1/4)[1-(\sin \theta'/n)^2]} r_g(\lambda)}{n^2 \left[ 1 - \left( \int_{\theta_0}^{\pi/2} \left[ t(\lambda) \right]^{(2/\cos \theta)} R_{10}(\theta) \sin 2\theta d\theta \right) r_g(\lambda) \right]}
\]
Note that in the specific case where the layer is clear (transmittance \( t = 1 \)), the Williams-Clapper reflectance becomes identical to the Clapper-Yule reflectance given by Eq. (230).
The substrat reflectance $r_g(\lambda)$ can thus be deduced from the measured spectral reflectance $R_w(\lambda)$ of the sample by using Eq. (231). Then, from the measured spectral reflectance $R_i(\lambda)$ of solid colorant patches, the normal transmittance $t_i(\lambda)$ of the colorant layer $i$ should be deduced numerically.

**Extension to halftone coloring layer**

When the coloring layer is not uniform but a halftone, the terms $T_{in}(\lambda)$, $T_{out}(\lambda)$ and $r_i(\lambda)$ expanded above for the different geometries should be replaced with the following ones:

\[
T_{in}(a_i, t_i, \lambda) = \sum_{k=1}^{8} a_k T_{in}(t_k, \lambda)
\]

\[
T_{ex}(a_i, t_i, \lambda) = \sum_{k=1}^{8} a_k T_{ex}(t_k, \lambda)
\]

\[
r_i(a_i, t_i, \lambda) = \sum_{k=1}^{8} a_k r_i(t_k, \lambda)
\]

Hence, the Williams-Clapper equation extended to halftones is

\[
R(\lambda) = r_g + \left[ \sum_{k=1}^{8} a_k T_{in}(t_k, \lambda) \right] \left[ \sum_{k=1}^{8} a_k T_{ex}(t_k, \lambda) \right] \left( 1 - \left[ \sum_{k=1}^{8} a_k r_i(t_k, \lambda) \right] r_g(\lambda) \right)
\]

and the formula (235) expanded for a measuring geometry based on collimated illumination at angle $\theta$ and a detector at angle $\theta'$ becomes

\[
R(\lambda) = \frac{T_{01}(\theta) T_{01}(\theta') \left( \sum a_k \left[ t_k(\lambda) \right]^{1/\sqrt{1-(\sin{\theta}/n)^2}} \right) \left( \sum a_k \left[ t_k(\lambda) \right]^{1/\sqrt{1-(\sin{\theta'}/n)^2}} \right) r_g(\lambda)}{n^2 \left( 1 - \left( \sum_{\theta_i=0}^{\pi/2} a_k \left[ t_k(\lambda) \right]^{2/\cos{\theta}} R_{i0}(\theta_i) \sin{2\theta} d\theta_i \right) r_g(\lambda) \right)}
\]

### 12.8. Four-ink halftones

The models presented above can be extended to four ink halftones in a straightforward manner, by increasing accordingly the number of spectral parameters and of inks spreading functions. The number of colorants (Neugebauer primaries) becomes $2^4 = 16$ instead of $2^3 = 8$ for 3-ink halftones. This yields 16 spectral parameters in the Spectral Neugebauer and Yule-Nielsen models.

The number of ink spreading functions is 4 (one per ink) in the BIS method and 32 (each of the four inks is superposed to the other three inks are printed at surface coverage 0 or 1) in the SDIS method. Each of them is established from three halftones in which one ink is at the nominal surface coverages 0.25, 0.5, and 0.75. The BIS and SDIS methods therefore need 12, respectively 96 printed halftones. One may also build the ink spreading functions by quadratic interpolation, which requires one halftone per function (ink printed at 0.5 surface coverage superposed with solid layers made of the other inks). In order to reduce this number, one may
print the ink at surface coverage 0.5 only and build the ink spreading functions by quadratic interpolation.

With 4-ink halftones, the spectral parameters and the effective ink surface coverages necessary to the building of the inks spreading functions are computed in the same manner as for 3-ink halftones. At prediction time, once the effective ink surface coverages of the four inks are computed, the surface coverages of the 16 colorants are provided by the 4-ink Demichel equations. They are then combined with the spectral parameters according to the model's general equation, which is similar for 3 or 4 inks.

12.9. Extensions of the models to transmittance and duplex prints

Recently, the Yule-Nielsen model, Clapper-Yule model and Williams-Clapper models have been extended to transmittance and enable predicting the spectral transmittance of papers whose two faces are printed with different halftones [104, 105], or the transmittance of stacks of halftone prints [106].
APPENDIX A – DESCARTES’ CONSTRUCTION FOR THE REFRACTION OF RAYS

Descartes' construction, illustrated by Fig. 48, allows determining the reflection and refraction directions of any incident ray on a flat interface by simple geometry [6, p. 101].

Let us consider a light ray coming from medium 1 and striking the interface at point A with an incidence angle $\theta_1$. Two circles $C_1$ and $C_2$ centered in A are drawn with respective radii $r_1$ and $r_2/n_2$. The line (MA), corresponding to the incident ray, intersects $C_1$ in points M and B; the tangent of $C_1$ issued from B intersects the interface at point B'. From B', two lines are drawn: the first one is tangent to the upper part of $C_1$ at point C; the second line is tangent to the lower part of $C_2$ at point D. The lines (AC) and (AD) correspond respectively to the reflected and transmitted rays.

![Fig. 48. Descartes' construction for the reflection and refraction of two light rays on an interface with relative refractive index $n_{21} = n_2/n_1 = 1.5$. The gray lines perpendicular to the rays represent the wave surfaces.](image)

This construction is justified by the Malus law, which asserts that the optical distance between two successive wave surfaces (represented by the gray lines orthogonal to the ray in Fig. 48) is the same for all the rays of a light pencil, even when the refractive index changes. Let us consider two rays propagating in parallel, represented by the lines (MA) and (M'B'). Point M
is the intersection of the first ray with circle $C_1$. Point $M'$, located on the second ray, belongs to the same wave front as $M$; $(MM')$ is therefore perpendicular to the two rays. For the construction, the distance between the two rays is selected in such manner that $M'B' = 2 MA$. Let us call $\tau$ the time necessary to travel the unit distance $MA = 1$ in medium 1. The speed of light in medium 1 is $v_1 = c/n_1$ where $c$ is the speed of light in vacuum. The travel time for a unit distance is $\tau = 1/v_1 = n_1/c$. During this time, the light travels a distance equal to $n_1/n_2$ in medium 2. At time zero, the two rays are respectively at points $M$ and $M'$. A time $\tau$, the first ray meets the interface at point $A$ where it splits into reflected and refracted rays. The second ray is in $A'$ on the same wave surface as $A$. At time $2\tau$, the second ray meets the interface at $B'$. In medium 1, the reflected ray has travelled a distance equal to 1. It is therefore located on circle $C_1$. It is also on the same wave surface as $B'$, therefore located at the point $C$ such that lines $(AC)$ and $(CB')$ are perpendicular. Regarding the refracted ray, it has travelled a distance $n_1/n_2$ in medium 2 and reached a point $D$ which again belongs to the same wave surface as $B'$. Therefore, $D$ is the tangency point between the straight line issued from $B'$ and the circle $C_1$ of radius $n_1/n_2$. Line $(AD)$ therefore corresponds to the direction of refraction.

**APPENDIX B – LAMBERTIAN REFLECTANCE OF INTERFACES**

**Integrating Fresnel reflectivity over the hemisphere**

The reflectance of a flat interface between media of indices $n_1$ and $n_2 > n_1$, illuminated by Lambertian light from the medium of index $n_1$, is expressed by the integral of Eq. (46), where the reflectivity for natural light, $R_{12}(\theta)$, is the average of the two components $R_{p12}(\theta)$ and $R_{s12}(\theta)$ expressed in equations (31) and (32)

$$r_{12} = \frac{1}{\pi} \int_{0}^{\pi/2} R_{12}(\theta) \sin 2\theta d\theta = \frac{1}{2} \int_{0}^{\pi/2} R_{s12}(\theta) \sin 2\theta d\theta + \frac{1}{2} \int_{0}^{\pi/2} R_{p12}(\theta) \sin 2\theta d\theta$$  \hspace{1cm} (237)

Let us denote $I_s$ the integral of $R_{s12}$, and $I_p$ the integral of $R_{p12}$. The easiest way to compute them goes through a first variable changing $u = \cos \theta$, then inversion of the Eq. $R_{12} = f(u)$, and finally integration according to the variable $\sqrt{R_{12}}$, denoted as $x$.

Let us first consider $I_s$. According to Eq. (34) and the notation $x = \sqrt{R_{s12}}$, we have

$$x^2 = \left[ \frac{u - \sqrt{n^2 - 1 + u^2}}{u + \sqrt{n^2 - 1 + u^2}} \right]^2$$  \hspace{1cm} (238)

By reversing Eq. (238), we obtain

$$u^2 = \left( \frac{n^2 - 1}{4} \right) (1 - x)^2 / x$$  \hspace{1cm} (239)

and by differentiating both members of (239), we obtain:
\[ 2udu = -\left(\frac{n^2-1}{4}\right) \frac{1-x^2}{x^2} dx. \]  

(I_s) becomes

\[ I_s = \int_0^{\pi/2} R_{s12}(\theta) \sin\theta d\theta = -2\int_0^1 R_{s12}(u)udu = \frac{n^2-1}{4} \int_{\pi/2}^1 x^2 \left(\frac{1-x^2}{x^2}\right) dx \]

where the limits of integration in the last integral correspond to \( \sqrt{R_{s12}(0)} = (n-1)/(n+1) \) and \( \sqrt{R_{s12}(\pi/2)} = 1 \). Computing this integral yields

\[ I_s = \frac{(n-1)(3n+1)}{3(n+1)^2} \]

Let us now calculate \( I_p \). In contrast with \( R_s \), \( R_p \) is not a monotone function of \( u \) in the interval \([0,1]\). It is decreasing between 0 and \( u_b = \cos \theta_b \), where \( \theta_b = \arctan(n) \) is the Brewster angle, then increasing between \( u_b \) and 1. It is therefore split into one integral on the interval \([0,u_b]\) and a second one on the interval \([u_b,1]\). The two integrals lie on similar equations, except a coefficient denoted as \( \gamma \) which is 1 in the interval \([0,u_b]\) and –1 in the intervals \([u_b,1]\). The square root of Eq. (34) yields

\[ x = \frac{x^2n^2u + \sqrt{n^2-1+u^2}}{n^2u + \sqrt{n^2-1+u^2}} \]

where \( x = \sqrt{R_{p12}} \). Note that the value of gamma in each interval comes from the constraint that \( 0 < x < 1 \). The inversion of (243) yields

\[ u^2 = \frac{1}{(n^2-1)} \cdot \frac{(1-x)^2}{(x+\gamma b)(bx+\gamma)} \]

with \( b = (n^2+1)/(n^2-1) \). By differentiating both members of (244), we obtain:

\[ 2udu = -\frac{4n^4}{(n^2-1)^3} \cdot \frac{1-x^2}{(x+\gamma b)^2 (bx+\gamma)^2} \]

The left member of Eq. (245), multiplied by \( R_{p12} \), is first integrated on the interval \([0,u_b]\), then on the interval \([u_b,1]\). Equivalently, the right member, multiplied by \( x^2 \), is first integrated on the interval \([1,0]\) or on the interval \([0,1]\) with introduction of a minus sign:

\[ 2\int_0^{u_b} R_s(u)udu = \frac{4n^4}{(n^2-1)^3} \int_0^1 \frac{1-x^2}{(x+b)^2 (bx+1)^2} dx = \frac{4n^4}{(n^2-1)^3} \left[ F_1(1) - F_1(0) \right], \]

then on the interval \([0,(n-1)/(n+1)]\)
\[ 2 \int_{u_b}^{1} R_p(u) \, du = \frac{4n^4}{(n^2 - 1)^3} \int_0^{\frac{\pi}{4n+1}} \frac{1 - x^2}{(x - b)^2 (bx - 1)^2} \, dx = \frac{4n^4}{(n^2 - 1)^3} \left[ F_{\frac{n-1}{n+1}}(0) - F_{\frac{n-1}{n+1}}(0) \right] \quad (247) \]

where \( F_{\gamma} \) denotes the antiderivative

\[ F_{\gamma}(x) = \int \frac{1-x^2}{(x + \gamma b)^2 (bx + \gamma)^2} \, dx \]
\[ = \frac{b^2}{(b^2 - 1)(b + \gamma x)} - \frac{1}{(b^2 - 1)b^3 (bx + \gamma)} - \frac{x}{b^2} + \gamma - \frac{b^4 \ln((x + \gamma b)^2) - \ln((bx + \gamma)^2)}{b^3 (b^2 - 1)} \quad (248) \]

The sum of the integrals (246) and (247), eased by the fact that \( F_1(0) + F_{-1}(0) = 0 \), provides

\[ I_p = \frac{4n^4}{(n^2 - 1)^3} \left[ F_1(1) + F_{-1}(\frac{n-1}{n+1}) \right] \]
\[ = 1 - \frac{4n^3(n^2 + 2n - 1)}{(n^2 - 1)(n^2 + 1)} \ln(n) + \frac{16n^4(n^4 + 1)}{(n^2 - 1)^2(n^2 + 1)} \ln(n) + \frac{2n^2(n^2 - 1)^2}{(n^2 + 1)^3} \ln(\frac{n-1}{n+1}) \quad (249) \]

Finally, the average of formulas (242) and (249) provides the Lambertian reflectance formula given in Eq. (47).

**Relationship between front and back Lambertian reflectances of an interface**

The reflectance formula (47) is valid when the Lambertian light comes from the medium with lowest refractive index. When it comes from the other medium, the reflectance is much higher because all rays with incident angle higher than the critical angle \( \theta_c = \arcsin(1/n) \) are totally reflected. There exists a simple relation between the two reflectances that we propose here to establish. As previously, we consider media of refractive indices \( n_1 \) and \( n_2 > n_1 \) (the relative index \( n = n_2/n_1 \) is therefore higher than one). Let us start from the integral expressing the reflectance at the side of medium 2 given by Eq. (49). We decompose it into two integrals on the intervals \([0, \theta_c]\) and \([\theta_c, \pi/2]\). In the interval \([0, \theta_c]\), relations (30), (37) and (42) yield the equality:

\[ R_{21}(\theta_2) \sin 2\theta_2 \, d\theta_2 = R_{12}(\theta_1) \frac{1}{n^2} \sin 2\theta_1 \, d\theta_1 \quad (250) \]

Instead of integrating the left member of (250) according to \( \theta_2 \) on the interval \([0, \theta_c]\), let us integrate the right member according to \( \theta_1 \) on the corresponding interval \([0, \pi/2]\) in which we retrieve the integral expressing reflectance \( r_{12} \) [see equations (46)]:

\[ \int_{\theta_1=0}^{\theta_c} R_{21}(\theta_2) \sin 2\theta_2 \, d\theta_2 = \frac{1}{n^2} \int_{\theta_1=0}^{\pi/2} R_{12}(\theta_1) \sin 2\theta_1 \, d\theta_1 = \frac{1}{n^2} \eta_{12} \quad (251) \]

In the interval \([\theta_c, \pi/2]\), the reflectivity is 1. One therefore has
\[
\int_{\theta_i}^{\pi/2} R_{21}(\theta_2) \sin 2\theta_2 d\theta_2 = \int_{\arcsin(1/n)}^{\pi/2} \sin 2\theta_2 d\theta_2 = 1 - \frac{1}{n^2}
\]

(252)

The sum of the two integrals provides the relation given in Eq. (50).

**Numerical values of interface reflectances and transmittances**

The reflectances and transmittances of flat interfaces whose numerical values are given in Table B.1 depend only the relative index \( n \). The three reflectances in question, \( R_{12}(0) \), \( r_{12} \) and \( r_{21} \) correspond respectively to the following three illumination geometries: directional illumination at normal incidence, Lambertian illumination from the medium with smallest index, and Lambertian illumination from the other medium. They are respectively given by the equations (33), (47) and (50). The three transmittances \( T_{12}(0) \), \( t_{12} \) and \( t_{21} \), defined by the same geometries as the reflectances, are simply given by: \( T_{12}(0) = 1 - R_{12}(0) \), \( t_{12} = 1 - r_{12} \), and \( t_{21} = 1 - r_{21} \).

Table B.1. Reflectance and transmittance of interfaces for different relative indices \( n \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( R_{12}(0) )</th>
<th>( T_{12}(0) )</th>
<th>( r_{12} )</th>
<th>( r_{21} )</th>
<th>( t_{12} )</th>
<th>( t_{21} )</th>
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<td>0.0170</td>
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</tr>
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AKNOWLEDGEMENTS

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